Tuesday, January 29 $*$ **Solutions** $*$ *Visualizing quadric surfaces*

1. Elliptic paraboloid: $z = Ax^2 + By^2$ (A, B have same sign)
   (a) The parabolas differ only by translation in the z-direction. In particular, they all curve in exactly the same way. To check this, note that setting $x = c$ in $z = x^2 + y^2$ gives $z = y^2 + c^2$.
   (b) If $A = 0$ or $B = 0$ our surface becomes a parabola extended out parallel to a coordinate axis. If $A = B = 0$ our surface becomes the plane $z = 0$. Neither of those surfaces are elliptic.
   (c) If A and B were both negative the surface would be a downward opening elliptic paraboloid contained entirely beneath the plane $z = 0$.

2. Hyperbolic paraboloid: $z = Ax^2 + By^2$ (A, B differ in sign)
   (a) The horizontal cross section given by $z = 0$ is a set of two crossing lines, which is not a hyperbola.
   (b) $y^2 - x^2 = -(x^2 - y^2)$ so the two surfaces would be mirrors of each other across the plane $z = 0$.

3. Ellipsoid: $\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$
   (a) To be a sphere we'd need $A^2 = B^2 = C^2$
   (b) The sliders cannot go to 0 since A, B and C are divisors in the equation.

4. Double cone: $z^2 = Ax^2 + By^2$
   (a) Setting z equal to a constant gives the equation for an ellipse, while setting x or y equal to a constant gives the equation for a hyperbola.
   (b) If $A = 0$ or $B = 0$ the equation yields a set of two intersecting planes.
   (c) The cross sections given by $x = 0$ or $y = 0$ are sets of two intersecting lines.

5. Hyperboloid of one sheet: $\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1$
   (a) The sliders don't go to 0 because A, B and C are divisors in the equation. When A, B, and C are very small, the hyperboloid is close to the double cone.
   (b) When $x = \pm A$, the equation reduces to $C^2y^2 = B^2z^2$, which describes two intersecting lines.
   (c) There must always be a hole through the hyperboloid, since when $z = 0$ our equation is $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, which describes a nontrivial ellipse (if (x, y) is in this ellipse, then so is $(-x, -y)$, and $(0, 0)$ does not satisfy this equation).

6. Hyperboloid of two sheets: $-\frac{x^2}{A^2} - \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$
   (a) The larger A and B get the smaller the terms $-\frac{x^2}{A^2}$ and $-\frac{y^2}{B^2}$ get, making the equation closer to one describing two planes.
   (b) There must always be a gap between the two sheets because the equation cannot be satisfied when $z = 0$.
   (c) These hyperboloids approach the double cone given by $z^2 = x^2 + y^2$. The algebraic way to see this is to rewrite the equation for the hyperboloid with $A = B = C$ as $z^2 = x^2 + y^2 + A^2$, and then argue that the final term becomes negligible as $A \to 0$.