Wed. April 24, 2019

Last time - practice with Stokes' Theorem

- please fill out the survey for finding a time for a review session.

Announcements.

Example.

$\vec{F} = \langle P, Q, R \rangle$ defined on all of S.

What can you say about $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$?

Divergence Theorem §16.9.

Assumptions. • $\vec{F}$ is a vector field on an open region $D \subset \mathbb{R}^3$, with continuous partial derivatives.

• $E \subset D$ is a "nice" solid

- we can integrate over E
- $S = \partial E$ is a piecewise smooth surface.

• orient S so that $\vec{n}$ points outward.

Theorem (Divergence Theorem)

$$\underbrace{\iiint_E \text{div}\,\vec{F}\, dV}_{\text{derivative}} = \underbrace{\iint_{\partial E} \vec{F} \cdot d\vec{S}}_{\text{boundary}} = \iint_{\partial E} \vec{F} \cdot \vec{n}\, dS$$

Recall: $\text{div}\,\vec{F} = P_x + Q_y + R_z$

Why? - The Fundamental Theorem of Calculus.

Example Go back to the surface S. Let E be the solid inside S, and assume $E \subset D$.

Use the divergence theorem to find $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iiint_E \text{div}(\text{curl}\,\vec{F})\, dV = \iiint_E 0\, dV = 0.$$

(we saw this earlier!)

Example: Verify the Divergence Theorem for $\vec{F} = \langle x, y, z \rangle$ and $E_r = \{x^2 + y^2 + z^2 \le r^2\}$ $(r > 0)$

that is
1) compute $\iiint_{E_r} \operatorname{div} \vec{F} \, dV$
2) compute $\iint_{\partial E_r} \vec{F} \cdot d\vec{S}$
3) check that they're equal, like the theorem says they should be.

1) $\operatorname{div} \vec{F} = 1 + 1 + 1 = 3.$

so $\iiint_{E_r} \operatorname{div} \vec{F} \, dV = 3 \iiint_{E_r} dV = 3$ (volume of ball of radius r)

$$= 3\left(\tfrac{4}{3}\pi r^3\right) = 4\pi r^3$$

2) $\iint_{\partial E_r} \vec{F} \cdot d\vec{S} = \iint_{\partial E_r} \vec{F} \cdot \vec{n} \, dS$

$$\vec{n} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = \frac{\langle x, y, z \rangle}{r}$$

$$\Rightarrow \vec{F} \cdot \vec{n} = \frac{1}{r} \langle x, y, z \rangle \cdot \langle x, y, z \rangle = \frac{1}{r}(x^2 + y^2 + z^2) = \frac{r^2}{r} = r$$

$\therefore \iint_{\partial E_r} \vec{F} \cdot d\vec{S} = \iint_{\partial E_r} r \, dS = r$ (surface area of sphere of radius r)

$$= r(4\pi r^2)$$
$$= 4\pi r^3.$$

3) They agree!

§ DIVERGENCE – physical meaning.

The Divergence Theorem says

$$\iiint_E \operatorname{div} \vec{F} \, dV = \iint_{\partial E} \vec{F} \cdot \vec{n} \, dS = \text{flux}$$

= amount of fluid leaving E in unit time.

For small E, around P: flux $\approx$ (volume E)($\operatorname{div}\vec{F}(P)$)

$\Rightarrow$ fluid leaves E $\iff$ $\operatorname{div}\vec{F} > 0$

fluid enters E $\iff$ $\operatorname{div}\vec{F} < 0$.

amount of fluid is constant $\iff$ $\operatorname{div}\vec{F} = 0$.

<u>Def.</u> P is a source if $\operatorname{div}\vec{F}(P) > 0$

P is a sink if $\operatorname{div}\vec{F}(P) < 0$.

<u>Example</u> Let $E = [-1,1]\times[-2,2]\times[-3,3]$

$\vec{F} = \langle 2xyze^{y^2}, -ze^{y^2}, (z+1)e^{y}\rangle$

How much liquid flows out of E in unit time?

↳ $\displaystyle\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$ by D.T.

$$\operatorname{div}\vec{F} = \cancel{4}\ 2yze^{y^2} - 2yze^{y^2} + e^{y} = e^{y}$$

$$\therefore \iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E e^{y}\, dV = \int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} e^{y}\, dz\, dy\, dx$$

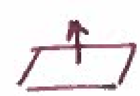
$$= \int_{-1}^{1} dx \int_{-2}^{2} e^{y}\, dy \int_{-3}^{3} dz$$

$$= (2)\left[e^{y}\right]_{-2}^{2}(6) = 12(e^{2}-e^{-2})$$

<u>Example:</u> How much flows across the sides and bottom of E?

<u>Strategy:</u>

$\partial E = S \cup S'$ where S' is the top.



$$\text{So } \underbrace{\iint_{\partial E} \vec{F}\cdot d\vec{S}}_{\text{know this}} = \underbrace{\iint_{S} \vec{F}\cdot d\vec{S}}_{\text{want this}} + \underbrace{\iint_{S'} \vec{F}\cdot d\vec{S}}_{\text{pretty easy to find this.}}$$

parametrize S' by $\vec{r}(u,v) = \langle u, v, 3\rangle$, $-1 \le u \le 1$, $-2 \le v \le 2$.

<u>check orientation:</u> $\vec{r}_u = \langle 1,0,0\rangle$, $\vec{r}_v = \langle 0,1,0\rangle$ $\Rightarrow$ $\vec{r}_u \times \vec{r}_v = \langle 0,0,1\rangle$

↳ points upward $\Rightarrow$ positively oriented.

So $\iint_{S'} \vec{F} \cdot d\vec{S} = \int_{-1}^{1}\int_{-2}^{2} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dv\, du$

$= \int_{-1}^{1}\int_{-2}^{2} \langle 2uv3e^{v^2}, \cdot 3e^{v^2}, 4e^v \rangle \cdot \langle 0,0,1 \rangle \, dv\, du$

$= \int_{-1}^{1}\int_{-2}^{2} 4e^v \, dv\, du$

$= 4 \cdot 2 \cdot (e^2 - e^{-2}) = 8(e^2 - e^{-2}).$

$\boxed{12}$ What is $\iint_S \vec{F} \cdot d\vec{S}$?

Example: Let $\vec{F} = \langle x,y,z \rangle$

How much liquid flows across

$\boxed{13}$ $S = \{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2\}$. ?