Last time - Polar coordinates.

[1] Calculate the area of the leaf (of the four-leafed rose)

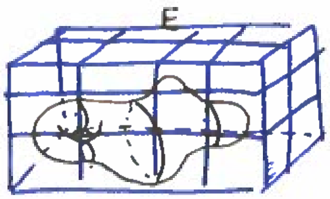
$$D = \{(r,\theta) \mid -\pi/4 \le \theta \le \pi/4, \quad 0 \le r \le \cos 2\theta\}.$$

TODAY: TRIPLE INTEGRALS (§ 15.7).

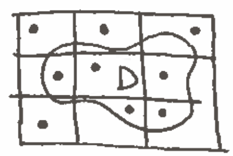
Assumption: all functions are continuous (where defined)

- Given $E \subset \mathbb{R}^3$ bounded and $f: E \to \mathbb{R}$ continuous, we define the integral of f over E similarly to how we defined double integrals

$$\iiint_E f \, dV \qquad\qquad \iint_D g \, dA.$$



- divide into subboxes
- choose random test points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$



- divide into subrectangles
- choose random test points

Geometric interpretations

1) $\iiint_E f \, dV$ = (average value of f on E) · (volume of E)

↳ 1b) when $f = 1$, average of f is 1

so $\iiint_E dV$ = volume of E.

2) Given a solid occupying the space defined by E, with density $\rho(x,y,z) \geq 0$ at $(x,y,z) \in E$:

Total mass: $m = \iiint_E \rho(x,y,z) \, dV$

Centre of mass: $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = \dfrac{\iiint_E x \rho(x,y,z) \, dV}{m}$ etc.

Moment of inertia about x-axis:

$$I_x = \iiint_E (y^2 + z^2)\,\rho(x,y,z)\,dV$$

(y^2+z^2) = (distance from point (x,y,z) to x-axis$)^2$

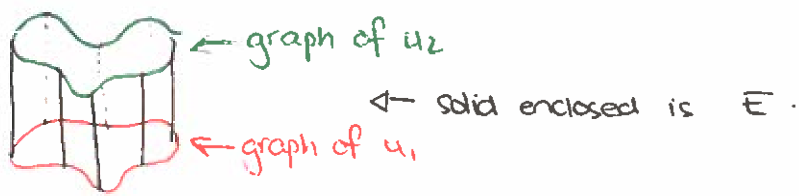
etc.

How do we calculate triple integrals?

* we have versions of Fubini's theorem for certain regions, (similar to regions of Type I and Type II in $\mathbb{R}^2$).

Suppose E can be written as

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}.$$



Then $$\iiint_E f\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f\,dz \right) dA. \quad (*)$$

- To find D — look for the shadow of E if a light shines down on it.

- To find u_1, u_2: given $(x_0,y_0) \in D$, $E \cap \{x = x_0, y = y_0\}$ is a vertical line segment with endpoints $u_1(x_0,y_0)$ and $u_2(x_0,y_0)$.

- if D is also nice, we can calculate (*):

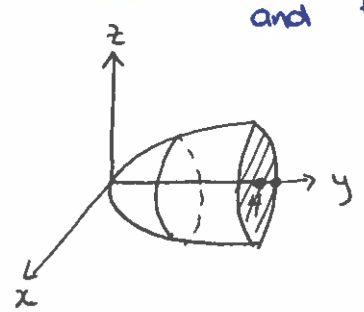
e.g. $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ (type I)

then $E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}$ and:

$$\iiint_E f\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

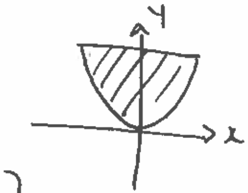
- Similar theorems for all six orders of dx, dy, dz.

Example: Let E be the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.



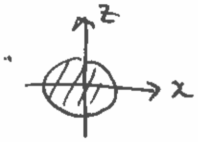
- Shadow on xy-plane:

$$D_1 = \left\{ (x,y) \,\middle|\, \begin{array}{l} -2 \le x \le 2 \\ x^2 \le y \le 4 \end{array} \right\}.$$



$$\Rightarrow E = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} \cdot\,(x,y) \in D \\ -\sqrt{y-x^2} \le z \le \sqrt{y-x^2} \end{array} \right\}.$$

- Shadow on (x,z) plane: $D_2 = \{ (x,z) \mid x^2 + z^2 \le 4 \}$.



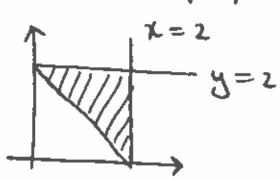
$$\Rightarrow E = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} \cdot\,(x,z) \in D_2 \\ \cdot\, x^2 + z^2 \le y \le 4 \end{array} \right\}.$$

Now let's use the second description to calculate the volume of E.
(See slides for solution.)

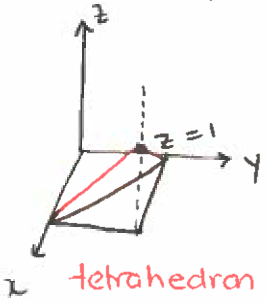
Another example. Let E be the region bounded by the four planes

- $x = 2$
- $y = 2$
- $z = 0$
- $x + y - 2z = 2$.

- Face on the xy plane ($z = 0$).



$x + y = 2$



tetrahedron

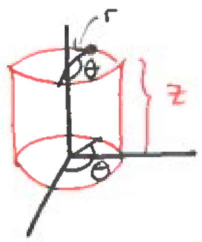
- Remaining vertex: $x = 2, y = 2,\ x + y - 2z = 2$

$$\Rightarrow z = 1$$

[2] Write $D = \left\{ (x,y) \,\middle|\, \begin{array}{l} ? \le x \le ? \\ ? \le y \le ? \end{array} \right\}$

[C] Write $E = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} (x,y) \in D \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}.$

CYLINDRICAL COORDINATES.

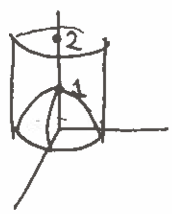


Given (x, y, z), write

$$x = r\cos\theta \qquad y = r\sin\theta, \qquad z = z.$$

Example: Let E be the region bounded by

- $x^2 + y^2 = 1$
- $z = 1 - x^2 - y^2$
- $z = 2$



- $x^2 + y^2 = 1$ i.e. $r^2 = 1$ i.e. $r = 1$
- $z = 2$
- $z = 1 - x^2 - y^2$ i.e. $z = 1 - r^2$

So
$$E = \left\{ (r, \theta, z) \;\middle|\; \begin{array}{l} \cdot\ 0 \le \theta \le 2\pi \\ \cdot\ 0 \le r \le 1 \\ \cdot\ 1 - r^2 \le z \le 2 \end{array} \right\}.$$