W. March 6 2019

Last time: Conservative vector fields I

[1] Find f s.t. $\nabla f = \vec{F} = \langle y\cos xy + 2xy,\ x\cos xy + 2e^{2y} + x^2 \rangle$

(See slides for solution)

- Theorem $\int_C \vec{F}\cdot d\vec{r}$ is independent of path & $\iff \int_C \vec{F}\cdot d\vec{r} = 0$ for all closed curves C.
- Theorem A. If D is ~~path~~ open and connected, then $\vec{F}$ is conservative $\iff \int_C \vec{F}\cdot d\vec{r}$ is path independent.

on slide.

Suppose $\vec{F} = \langle P, Q \rangle$ on $D \subset \mathbb{R}^2$, where P, Q have continuous first order partial derivatives.

Theorem: If $\vec{F}$ is conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

proof $\vec{F}$ conservative $\Rightarrow \exists f$ s.t.

$\vec{F} = \nabla f$, so $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$.

So $$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \underset{\text{Clairaut's theorem}}{=} \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}. \qquad \square$$

Is the converse true? If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, is $\vec{F}$ conservative?

[Slide]
- Not quite
- Need assumptions on the domain D.

- Examples: $\vec{F} = \langle y, -x \rangle$ - $\frac{\partial P}{\partial y} = 1$, $\frac{\partial Q}{\partial x} = -1 \Rightarrow$ Not conservative.

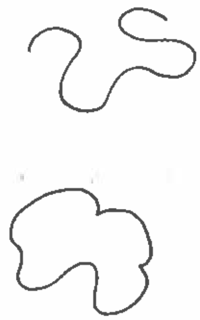
  $\vec{F} = \langle y, 1 \rangle$. $\frac{\partial P}{\partial y} = 1$, $\frac{\partial Q}{\partial x} = 0 \Rightarrow$ Not conservative.

(Much easier than checking for path independence!)

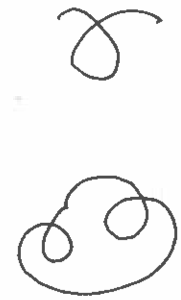
Definition: Let a curve C be parametrized by $\vec{r}(t)$, $a \le t \le b$.

C is simple if it doesn't intersect itself, except possibly for $\vec{r}(a) = \vec{r}(b)$.

Examples

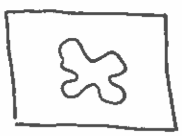


Simple   Not simple



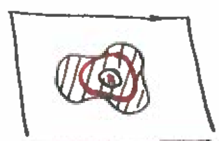
A set $D \subset \mathbb{R}^2$ is simply connected if

- it is connected
- it doesn't contain any holes
  - ↳ more precisely, each simple closed curve in D ~~contains~~ only encloses points in D.



Simply connected

Not connected



simple closed curve - contains points not in D

connected, but not simply connected.

[1] Which ~~sets~~ are open and simply connected?

Theorem (B) If $D \subset \mathbb{R}^2$ is open and simply connected, with $\vec{F} = \langle P, Q \rangle$ defined on D with continuous first order partial derivatives.

Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F}$ is conservative.

- We will prove this later (by showing $\vec{F}$ is path independent).

Example: Is $\vec{F}(x,y) = \langle y, x+y \rangle$ conservative?

Short answer: Yes! (B) applies & $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$.

Longer answer: Yes - $\vec{F} = \nabla f$, $f = (xy + \frac{1}{2}y^2)$.

Summary of approaches:

Given $\vec{F}$ on D, is it conservative?

* Method B — use this if $D \subset \mathbb{R}^2$ is open & simply connected.

- is $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$?
  - Yes $\Rightarrow$ conservative
  - No $\Rightarrow$ Not conservative.

* Method A - use this if you can't use (B):
(e.g. $D \subset \mathbb{R}^3$; $D \subset \mathbb{R}^2$ not simply connected;
No formulas for P, Q).

(is D open & connected?)

and is $\vec{F}$ path independent?

- Yes $\Rightarrow$ Conservative
- No $\Rightarrow$ Not conservative.

[Method A' : is $\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths].

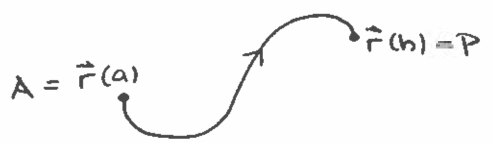
$\boxed{11}$. Let $\vec{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$

§ CONSERVATION OF ENERGY.

A force $\vec{F}$ acts on a mass m.

Let $\vec{r}(t)$ be its position at time t, $t \in [a,b]$.

$\Rightarrow$ $\vec{F} = m\vec{a} = m\,\vec{r}\,''(t)$.



Kinetic energy at time t:

$$K(t) = \frac{1}{2} m\, |\vec{r}\,'(t)|^2$$

Claim 1: The work done by $\vec{F}$ is $K(b) - K(a)$.

proof. $\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$= \int_a^b m\,\vec{r}\,''(t) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \frac{m}{2} \frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) dt = \frac{m}{2} \int_a^b \frac{d}{dt} |\vec{r}\,'(t)|^2\, dt$$

$$= \frac{m}{2}\Big[|\vec{r}\,'(t)|^2\Big]_a^b = K(b) - K(a).$$

~~222.~~ 121.4

Now suppose $\vec{F}$ is conservative with potential f.

The potential energy of the particle at $Q = (x, y, z)$ is

$$P(Q) = -f(Q).$$

FTLI $\Rightarrow$ $W = \int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$

$$= P(A) - P(B).$$

Comparing with Claim 1:

$$P(A) - P(B) = K(b) - K(a)$$

i.e. $P(A) + K(a) = P(B) + K(b)$.

$\Rightarrow$ Total amount of energy is conserved. when a particle is acted on by a conservative vector field.