Integrating functions over curves

Recall that for a (smooth) curve C parametrized by a vector-valued function $\mathbf{r}$ over an interval $[a, b]$, and for a function $f : C \to \mathbb{R}$, we have

$$\int_C f \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt.$$

This formula works whether C is a plane curve ($\mathbf{r} : [a, b] \to \mathbb{R}^2$) or a space curve ($\mathbf{r} : [a, b] \to \mathbb{R}^3$).

Compute $\int_C x^2 z ds$ where C is the line segment from $(0, 6, -1)$ to $(4, 1, 5)$.

(a) $\frac{56}{3}\sqrt{77}$

(b) $\frac{14}{3}\sqrt{77}$

(c) $\frac{56}{3}\sqrt{15}$

(d) $\frac{14}{3}\sqrt{15}$

Correct answer: (a)

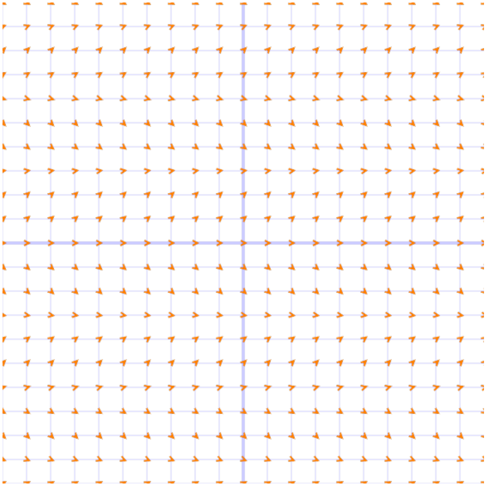
Announcements

- **Midterm 2** is on Tuesday, March 12 at 7pm.
  - Deadline to request a spot in the conflict exam is next Tuesday, March 5.
- **Register your i-clicker!** Deadline is this Saturday, March 2, at 5pm.
  - Check on Moodle: if you do not see any i-clicker grades, your registration has not gone through. Email me with your name, i-clicker number, and netid.
- **Thanks for your feedback.**
  - Changes: bigger chalk, more examples, more slides when possible.
  - Please continue to provide feedback (by email or anonymously e.g. through your TA).

An example of a vector field

https://earth.nullschool.net/

Matching a vector field with its plot



(a) $\mathbf{F}(x, y) = \langle \sin(x), 1 \rangle$

(b) $\mathbf{F}(x, y) = \langle 1, \sin(y) \rangle$

(c) $\mathbf{F}(x, y) = \langle 1, \cos(y) \rangle$

(d) $\mathbf{F}(x, y) = \langle \sin(y), 1 \rangle$

(e) I don't know how

Correct answer: (b)

Practice with integrating vector fields

Let $\mathbf{r}(t) = \langle t, t^2 \rangle, t \in [0, 1]$, and let $\mathbf{F}(x, y) = \langle y, x \rangle$. Sketch the curve and vector field. What can you say about $\int_C \mathbf{F} \cdot d\mathbf{r}$?

(a) It's positive.

(b) It's negative.

(c) It's zero.

(d) It's not defined.

(e) I don't know how to say anything about it.

Correct answer: (a)

Practice with integrating vector fields

Let $\mathbf{r}(t) = \langle t, t^2 \rangle$, $t \in [0, 1]$, and let $\mathbf{F}(x, y) = \langle y, x \rangle$ (as on the previous slide).

- $\mathbf{F}(\mathbf{r}(t)) = \langle t^2, t \rangle$.
- $\mathbf{r}'(t) = \langle 1, 2t \rangle$.

It follows that

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle t^2, t \rangle \cdot \langle 1, 2t \rangle dt \\ &= \int_0^1 3t^2 dt \\ &= [t^3]_0^1 = 1. \end{aligned}$$

(Note that this is positive.)

Practice with integrating vector fields

Let C be parametrized by $\mathbf{r}(t) = \langle t, 2t \rangle$, $t \in [0, 1]$. Let $\mathbf{F}(x, y) = \langle 1, 2y \rangle$.

What is $\int_C \mathbf{F} \cdot d\mathbf{r}$?

(a) 9

(b) 5

(c) 0

(d) 20

(e) I don't know what to do.

(If you're done, sketch the curve and the vector field, and check whether your answer is a reasonable one.)

Correct answer: (b)

Solution:

Let C be parametrized by $\mathbf{r}(t) = \langle t, 2t \rangle$, $t \in [0, 1]$.
Let $\mathbf{F}(x, y) = \langle 1, 2y \rangle$.

- $\mathbf{F}(\mathbf{r}(t)) = \langle 1, 4t \rangle$.
- $\mathbf{r}'(t) = \langle 1, 2 \rangle$.

$$
\begin{aligned}
\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 1, 4t \rangle \cdot \langle 1, 2 \rangle dt \\
&= \int_0^1 1 + 8t \; dt \\
&= [t + 4t^2]_0^1 \\
&= 5.
\end{aligned}
$$