Last time: The chain rule: $h(s,t) = f(x(s,t), y(s,t))$

$$\Rightarrow \frac{\partial h}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} \; ; \; \frac{\partial h}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}.$$

2

§ DIRECTIONAL DERIVATIVES.

Let $\vec{u} = \langle a,b\rangle$ be a unit vector; $f: \mathbb{R}^2 \to \mathbb{R}$

Definition: the directional derivative of f at (x_0,y_0) in the direction $\vec{u}$ is the rate of change of f at (x_0,y_0) as we move in the direction $\vec{u} = \langle a,b\rangle$.

$$\Rightarrow D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(\langle x_0,y_0\rangle + h\vec{u}) - f(x_0,y_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb)}{h}.$$

Rmk: When $\vec{u} = \langle 1,0\rangle$, $D_{\vec{u}}f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0)}{h} = \frac{\partial f}{\partial x}(x_0,y_0)$

Likewise, $D_{\langle 0,1\rangle} f(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0)$.

But $D_{\langle -1,0\rangle} f(x_0,y_0) = -\frac{\partial f}{\partial x}(x_0,y_0)$.

Calculating using the chain rule:

$D_{\vec{u}} f(x_0,y_0) = \frac{dg}{dt}(0)$, where $g(t) = f(x(t), y(t))$
$x(t) = x_0 + ta$, $y(t) = y_0 + tb$.

$$\boxed{= \frac{\partial f}{\partial x}(x_0,y_0)\, a + \frac{\partial f}{\partial y}(x_0,y_0)\, b.}$$

Example: $f(x,y) = x^2 + y^3$; $\vec{u} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$.

- Find $D_{\vec{u}} f(3,1)$.

$f_x(x,y) = 2x \Rightarrow f_x(3,1) = 6$

$f_y(x,y) = 3y^2 \to f_y(3,1) = 3$.

- $D_{\vec{u}} f(3,1) = f_x(3,1)\cdot\frac{3}{5} + f_y(3,1)\left(-\frac{4}{5}\right) = \frac{18-12}{5} = \frac{6}{5}$.

In higher dimensions: $\vec{u} = \langle u_1, u_2, \dots, u_n\rangle$ - unit vector

$f: \mathbb{R}^n \to \mathbb{R}$ differentiable at $P \in \mathbb{R}^n$.

↳ $D_{\vec{u}} f(P) = \frac{\partial f}{\partial x_1}(P)\, u_1 + \cdots + \frac{\partial f}{\partial x_n}(P)\, u_n.$

Example: let $T(x,y,z) = \cancel{T}\ xyz$ be the temperature at location (x,y,z)

Suppose there is a butterfly flying around. Right now it is at

$\boxed{2}$ point $P = (1,2,3)$ and has velocity $\vec{v} = \langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle$.

§ GRADIENTS.

Definition: $\nabla f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle \quad : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

↳ so $D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u}$.

Example $T(x,y,z) = xyz$.

- $\nabla T = \langle yz, xz, xy \rangle$

$\Rightarrow \nabla T(1,2,3) = \langle 6, 3, 2 \rangle$.

Gradient helps us to answer the following questions:

1) In what direction is the rate of increase of f largest at a point P?

2) What is the maximum rate (i.e. value $D_{\vec{u}}(P)$)

check all unit vectors $\vec{u}$.

Note:

$$D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u}$$
$$= |\nabla f(P)| \cdot |\vec{u}| \cdot \cos\theta$$
$$= |\nabla f(P)| \cos\theta.$$

this is maximal when $\theta = 0$

i.e. $\nabla f(P)$ and $\vec{u}$ point in the same direction

(Careful: $\vec{u} = \frac{\nabla f(P)}{|\nabla f(P)|}$ - norm 1)

and the maximum value is $|\nabla f(P)|$

Likewise the minimum value occurs for $\vec{u} = -\frac{\nabla f(P)}{|\nabla f(P)|}$

and is $-|\nabla f(P)|$.

What if $\nabla f(P) = 0$?? → we'll see in a minute.

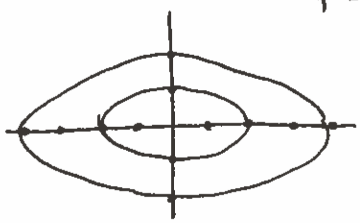
$\boxed{1}$ Question: $T(x,y,z) = xyz$. $P = (1,2,3)$.

What is the maximum rate of increase of T at P?

Example: $f(x,y) = 1 - x^2 - 4y^2$

Level sets: $f = 0$: $1 - x^2 - 4y^2 = 0 \iff x^2 + 4y^2 = 1$

$f = 3$: $1 - x^2 - 4y^2 = -3 \iff x^2 + 4y^2 = 4$



$\boxed{2}$ Draw the vector $\nabla f(1,0)$, with its tail at the point $(1,0)$.

Where is the max value of the function f?

Does $\nabla f(1,0)$ point towards this point?

Is it tangent to the contour line through $(1,0)$?

What about the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$?

Show illustration. $\nabla f(0,0) = \langle 0, 0 \rangle$.

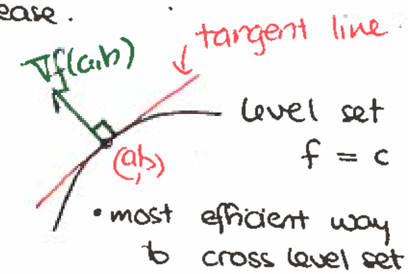
$\Rightarrow$ all directional derivatives at $(0,0)$ are 0.

Morals: 1) ∇f points in the direction of fastest increase.

2) ∇f is orthogonal to the level set

3) A max/min occurs when

$\nabla f(P) = \langle 0, 0 \rangle$



4) $\nabla f(P)$ does not necessarily point towards the maximum, because it only knows local behaviour of the function.

But ∇f can be used to find maxima/minima:

• travel in small steps:

· Start at P

· travel a small distance in direction $\nabla f(P)$.

· Call this point Q.

· travel in direction $\nabla f(Q)$.

Repeat.