MATH 595 Tuesday 6 March

Serre duality for projective spaces

(1) **Chapter III, Exercise 5.7.**

(Actually this question is still about ideas from Section 5. You're going to get a lot of practice with the cohomological criterion for a line bundle to be ample.)

Let X be a proper scheme over a noetherian ring A. Let $\mathscr{L}$ be a line bundle on X.

(a) Suppose that $Y \subset X$ is a closed subscheme, and denote the closed embedding by i. Prove that if $\mathscr{L}$ is ample on X, then $i^*\mathscr{L}$ is ample on Y.

(b) Show that $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{\mathrm{red}} = \mathscr{L} \otimes \mathcal{O}_{X_{\mathrm{red}}}$ is ample on X_{red}.

(c) Suppose that X is reduced. Prove that $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on each irreducible component X_i of X.

(d) For this question, you will use the following lemma (it was an exercise in section 3 that we skipped):

Lemma 1. *Let $f : X \to Y$ be a finite surjective morphism of degree m between integral schemes X and Y. Then for any $\mathscr{F} \in \mathrm{Coh}(Y)$ there exists a sheaf $\mathscr{G} \in \mathrm{Coh}(X)$ and a homomorphism $u : f_*(\mathscr{G}) \to \mathscr{F}^{\oplus m}$ such that u is a generic isomorphism.*

With this in mind, now assume that f is any finite surjective morphism from X to a scheme Y. Let $\mathscr{M}$ be a line bundle on Y. Prove by induction on the dimension of X and Y that $\mathscr{M}$ is ample if and only if $f^*\mathscr{M}$ is ample.

(2) **Chapter III, Exercise 7.2.**

Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k. Recall that (in the exercises on 15 February) we defined for $\mathscr{G} \in \mathrm{QCoh}(Y)$ a quasi-coherent $\mathcal{O}_X$-module $f^!\mathscr{G}$, with the defining property

$$f_*f^!(G) = \mathscr{H}om_Y(f_*(\mathcal{O}_X), \mathscr{G}).$$

We saw that it comes equipped with a natural morphism $f_*f^!\mathscr{G} \to \mathscr{G}$.

With this in mind, suppose that (ω_Y°, t) is a dualizing sheaf on Y. Prove that $f^!\omega_X^\circ$ is a dualizing sheaf for X.