MATH 402 Review for November 26–30

Topics: Area and defect in hyperbolic geometry; regular tilings in Euclidean and hyperbolic geometry; complex numbers and the extend complex plane; stereographic projection and an isomorphism between the Klein model and the Poincaré model.

These were covered in lecture, and in the project (7.7). This material also appears on the optional homework, homework 11 (highly recommended for exam preparation!).

1. **Recall from earlier in the term:** Area satisfies three axioms: the area of a triangle is positive; two equivalent sets have the same area; and the area of a disjoint union of sets is the same as the sum of the areas of each of the sets. (Recall that two polygonal sets are *equivalent* if we can cut them up into polygonal (or equivalently triangular) pieces, and then match the pieces.)

   Review the definition of defect (for a triangle, quadrilateral, or polygon with arbitrarily many sides).

2. **Things to know about defect and area:**

   (a) You can show that defect is additive, so it satisfies the third axiom of area. (See homework 11 for examples.)

   (b) We proved that two hyperbolic triangles have the same defect if and only if they are equivalent. (So defect also satisfies the second axiom of area.)

   (c) We proved last week that the defect of a hyperbolic triangle is always positive, so defect satisfies the first axiom area as well.

   (d) It follows that any way of defining area in hyperbolic geometry which satisfies all three axioms will be closely related to defect: there will be some real number k such that for any triangle $\triangle ABC$, we have

   $$\text{area}(\triangle ABC) = k^2 \text{defect}(\triangle ABC).$$

   (In particular, the area of $\triangle ABC$ is *at most* $k^2 180$. This is not true at all in Euclidean geometry! You can make triangles in Euclidean geometry with area as large as you like.)

3. **Things to know about regular tilings:**

   (a) A tiling is of *type* (n, k) if it consists of n-gons as tiles meeting k at each vertex.

   (b) In Euclidean geometry, a tile of type (n, k) exists if and only if

   $$\frac{1}{n} + \frac{1}{k} = \frac{1}{2}.$$

   The only integer solutions are $(3, 6)$ (a tiling by equilateral triangles); $(4, 4)$ (a tiling by squares); and $(6, 3)$ (a tiling by hexagons).

   (c) However, in hyperbolic geometry, because angle sums are smaller than in Euclidean geometry, the equation becomes

   $$\frac{1}{n} + \frac{1}{k} < \frac{1}{2}.$$

   (Know the proof of this statement.) You prove on the project that there are infinitely many such solutions, and you learn how to construct the tiles for an (n, k)-tiling.

4. **Things to know complex numbers, the extended complex plane, and the isomorphism F.**

   (a) The complex plane is given by $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$; we have $\mathbb{C} \cong \mathbb{R}^2$. Know how to add and multiply complex vectors, take the complex conjugate, and determine the modulus.

   (b) The *extended complex plane* $\overline{\mathbb{C}}$ is what we get when we add a point ∞. Stereographic projection gives us an isomorphism between $\overline{\mathbb{C}}$ and the two dimensional sphere. Draw pictures illustrating how the map works. See the homework for practice deriving the formula of $\pi : S^2 \to \overline{\mathbb{C}}$. (Important: notice that I made a typo on the board in lecture on Friday! I initially gave the formula for stereographic projection's inverse π^{-1} correctly, but later I made a sign error. The correct denominator is $|z|^2 + 1$ (not $|z|^2 - 1$).)

   (c) Combining stereographic projection and vertical projection (from the disk at $Z = 0$ to the bottom half of the sphere) gives us a function F from the Klein disk to the Poincaré disk. We proved that F sends Klein lines to Poincaré lines. We're also close to proving that F sends perpendicular Klein lines to perpendicular Poincaré lines.

Practice Questions

1. To get used to the idea of equivalence, here are some exercises in Euclidean geometry from the textbook:

   (a) Exercise 2.4.4

   (b) Exercise 2.4.6

   We have been using the fact (without proof) that if a shape A is equivalent to a shape B, and B is equivalent to C, then A is equivalent to C. Draw an example of A, B, and C, and check that this is true. (The tricky part is that the pieces that you cut B into to compare it to A can be different from the pieces you cut it into to compare it to C.)

   Now look at the example of two triangles with the same defect and one side congruent: in this notation, the two triangles will be A and C, and we construct a Saccheri quadrilateral B which is equivalent to both of them. What pieces do we need to cut B into to compare it to A? What about to C? What pieces should we cut A into to compare it directly to C?

2. Find lots more practice questions on the optional homework, Homework 11.