Exercise 1.

(a) Write down the definition of a group. [5 points]

A group G consists of a set of elements and a binary operation "$\cdot$" that relates two elements of the group to a third.

The axioms for a group are:

(A1) For all elements x and y we have $x \cdot y \in G$.

(A2) The binary operation is associative. $\forall\ x, y, z \in G$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

(A3) There is a special element $e \in G$ s.t. $x \cdot e = x \quad \forall\ x \in G$.

The element e is called the identity of G.

(A4) Given $x \in G$, there is an element $x^{-1} \in G$ s.t. $x \cdot x^{-1} = e$.

The element x^{-1} is called the inverse to x

(b) Let G be the set of invertible functions $\mathbb{R} \to \mathbb{R}$, with $\circ$ given by composition of functions. [10 points]

Prove that this is a model for the above axiomatic system.

(i): Associativity:

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x)))$$

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))) \quad \forall\ x \in \mathbb{R}.$$

$\Rightarrow (f \circ g) \circ h = f \circ (g \circ h)$ as functions $\mathbb{R} \to \mathbb{R}$.

(ii) Identity:

let $e = id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $x \longmapsto x$. It is an invertible function,

and if $f \in G$ is any element,

$$(f \circ e)(x) = f(e(x)) = f(x) \quad \forall\ x \in \mathbb{R}$$

$\Rightarrow f \circ e = f$ as required.

(iii) Inverses:

each element $f \in G$ has an inverse $f^{-1} \in G$ by definition of G.

iv: Closure under $\circ$ (Axiom A1)

- If $f, g \in G$, it is clear that $f \circ g: \mathbb{R} \to \mathbb{R}$.
  But we need to check that $f \circ g$ has an inverse.
- Note that f, g each have inverses f^{-1} and g^{-1}

Claim: $g^{-1} \circ f^{-1}$ is an inverse for $f \circ g$

$$
\begin{aligned}
(g^{-1} \circ f^{-1}) \circ (f \circ g) &= ((g^{-1} \circ f^{-1}) \circ f) \circ g && \text{by associativity} \\
&= (g^{-1} \circ (f^{-1} \circ f)) \circ g && \text{by associativity} \\
&= (g^{-1} \circ e) \circ g && \text{since } f^{-1} \circ f = e \\
&= g^{-1} \circ g && \text{since } g^{-1} \circ e = g^{-1} \\
&= e
\end{aligned}
$$

Likewise $(f \circ g) \circ (g^{-1} \circ f^{-1}) = e$

Exercise 2:

(a) Let G be a group, and suppose $x, y, z \in G$ satisfy $x \cdot z = y \cdot z$. Show that $x = y$. [5 points]

- $x \cdot z = y \cdot z \Rightarrow (x \cdot z) \cdot z^{-1} = (y \cdot z) \cdot z^{-1}$
- But

$$
\begin{aligned}
(x \cdot z) \cdot z^{-1} &= x(z \cdot z^{-1}) && \text{by associativity} \\
&= x \cdot e && \text{by property of inverses} \\
&= x && \text{by property of identity}
\end{aligned}
$$

and likewise $(y \cdot z) \cdot z^{-1} = y$

$\therefore$ $x = y$ as claimed

(b) If $x \in G$, $x \cdot e = e \cdot x$. [5 points]

By (a), it suffices to show that $(x \cdot e) \cdot x^{-1} = (e \cdot x) \cdot x^{-1}$

But

$$
\begin{aligned}
(x \cdot e) \cdot x^{-1} &= x \cdot x^{-1} && \text{by property of } e \\
&= e
\end{aligned}
$$

and

$$
\begin{aligned}
(e \cdot x) \cdot x^{-1} &= e(x \cdot x^{-1}) && \text{by associativity} \\
&= e \cdot e && \text{by property of inverses} \\
&= e && \text{by property of } e.
\end{aligned}
$$

$\therefore$ the claim holds.

(c) The identity element is unique. [5 points]

proof: Suppose $\exists$ an element $e' \in G$ s.t $x \cdot e' = x \quad \forall \; x \in G$

we want to show that $e' = e$

- $e \cdot e' = e$ by definition of e'

But also $e \cdot e' = e' \cdot e$ by (b)

$= e'$ by definition of e.

$\therefore \; e = e'$ as claimed.

Exercise 3: S, $P \subset S \times S$ s.t.

(1) if $(a,b) \in \cancel{S \times S}. P \quad (b,a) \notin \cancel{S \times S} \; P$

(2) if $(a,b), (b,c) \in \cancel{S \times S} \; P \quad (a,c) \in \cancel{S}. P$.

(a) Let $S_1 = \{1, 2, 3, 4\}$, $P_1 = \{(1,2), (2,3), (1,3)\}$ [5 points]

Is this a model for the system?

Yes. We can show that the axioms hold.

(1). if $(a,b) \in P$ we need to show $(b,a) \notin P$..

- $(a,b) = (1,2) \rightsquigarrow (2,1) \notin P$
- $(a,b) = (2,3) \rightsquigarrow (3,2) \notin P$.
- $(a,b) = (1,3) \rightsquigarrow (3,1) \notin P$.

So axiom 1 holds.

(2). we need to look at pairs $(a,b), (b,c) \in P$,.

The only such pair is $(1,2)$ and $(2,3)$.

the axiom says that $(1,3)$ should be in P_1, and it is.

(b) Let $S_2 = \mathbb{R}$, $P_2 = \{(x,y) \mid x < y\}$. Is this a model? [5 points]

Yes. We check the axioms again.

(1): if $(a,b) \in P_2$, then $a < b$, so $b \nless a$, and hence $(b,a) \notin P_2$

(2) if $(a,b), (b,c) \in P_2$, then $a < b$, $b < c$,

so $a < c$, and hence $(a,c) \in P_2$.

(c) Use this to argue that the axiomatic system is not complete. [5 points]. 4

we can add a third axiom which is independent from the first two and also consistent with them.

e.g. (A3) there are only finitely many elements of S.

* the model (S_1, P_1) satisfies this axiom but the model (S_2, P_2) does not