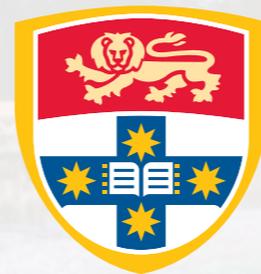


# 8th Australia-New Zealand Mathematics Convention

## Melbourne 8-12 December, 2014

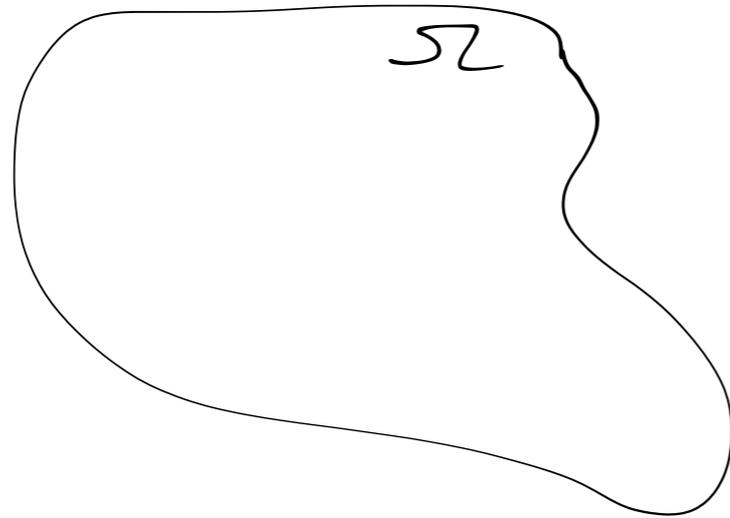
### Uniform convergence of solutions to Robin & Neumann boundary problems on domains with shrinking holes

Daniel Hauer



THE UNIVERSITY OF  
SYDNEY

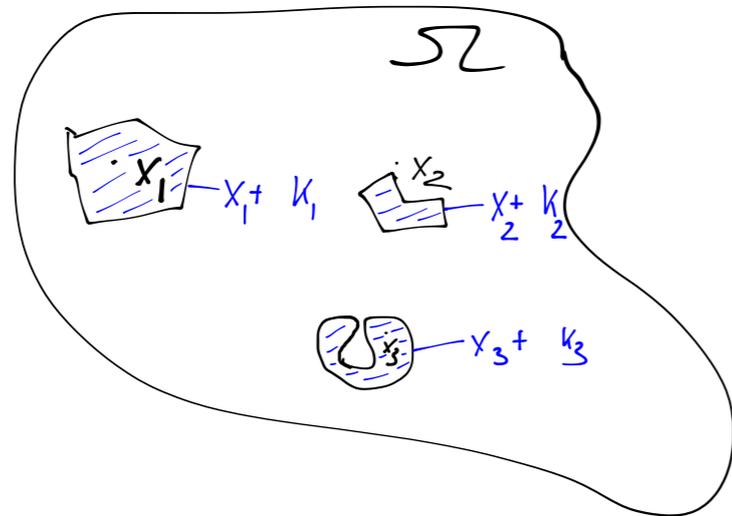
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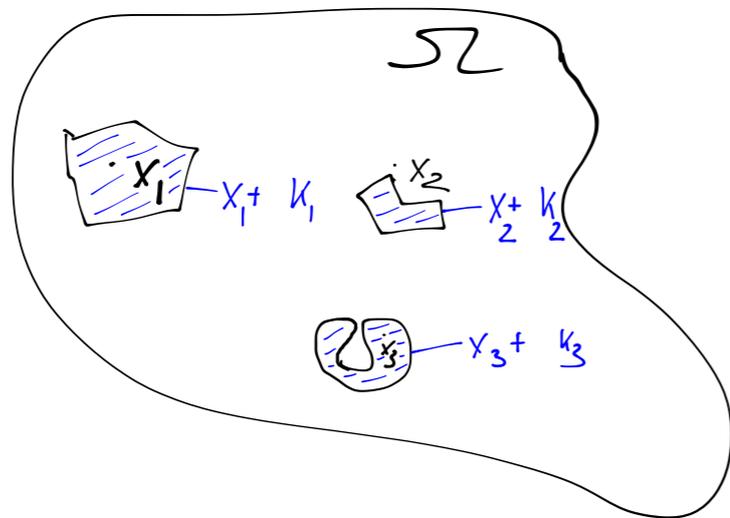
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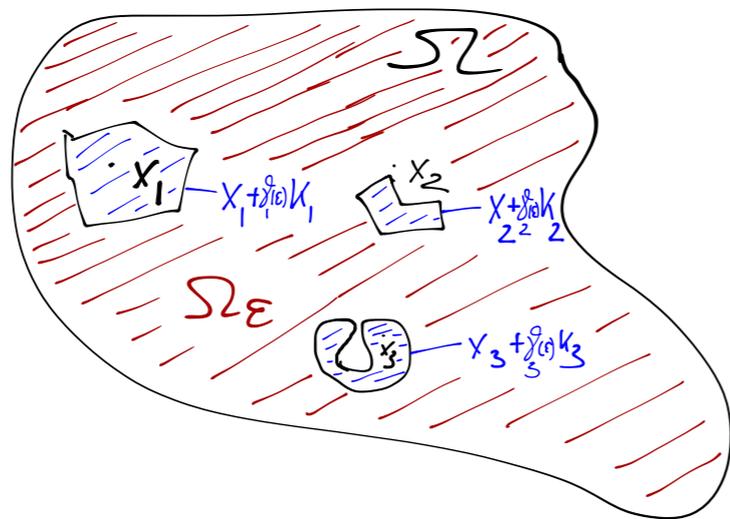
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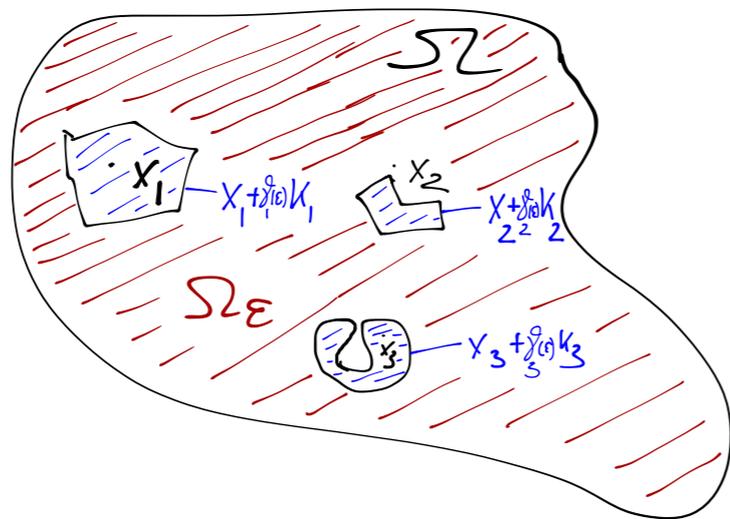


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$\uparrow$  shrinking function  
 $\delta_j: (0, 1] \rightarrow (0, 1]$   
s.t.  $\lim_{\epsilon \rightarrow 0^+} \delta_j(\epsilon) = 0$ .

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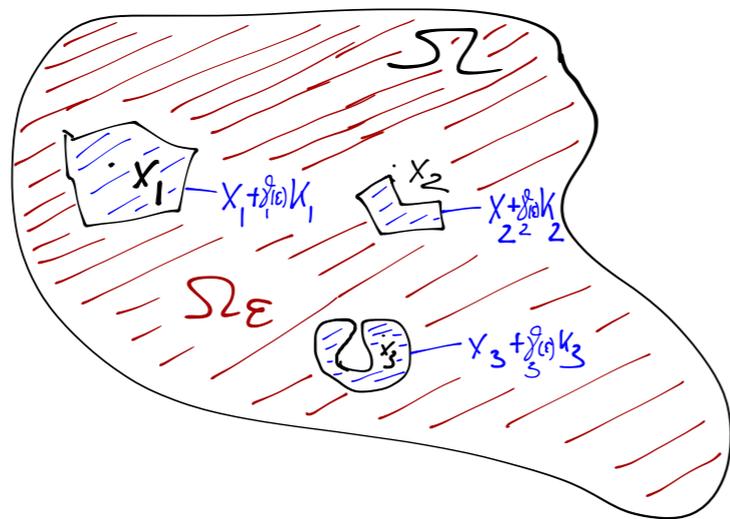
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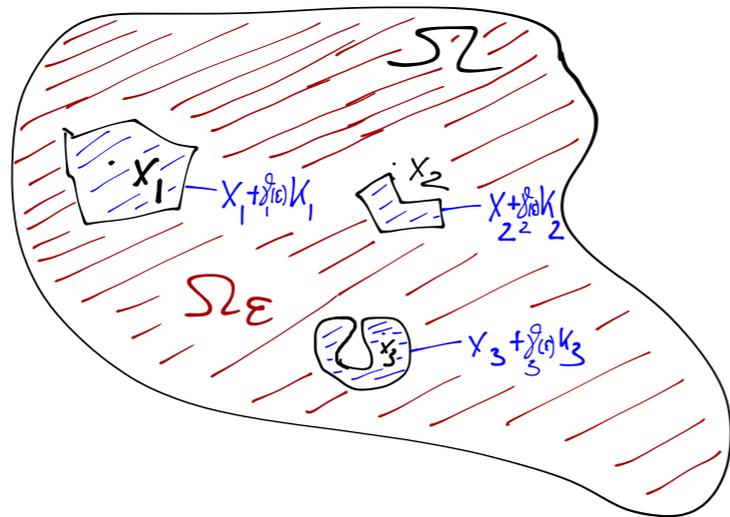
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& set  $\Omega_\epsilon := \Omega \setminus H_\epsilon$ .

In  $d=2$ , let  $\partial K_j$  be a closed curve & of class  $C^{1,0}$  for  $\theta \in (a, 1)$ .



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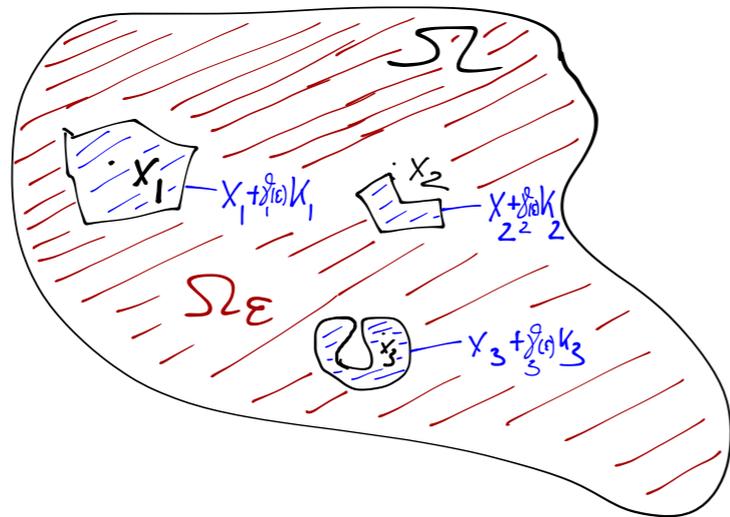


For  $f_\varepsilon \in L^p(\Omega_\varepsilon)$  let  $u_\varepsilon$  be the weak solution of

$$(1) \begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \partial H_\varepsilon \end{cases}$$



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and for  $f \in L^p(\Omega)$  let  $u$  be the unique weak solution of

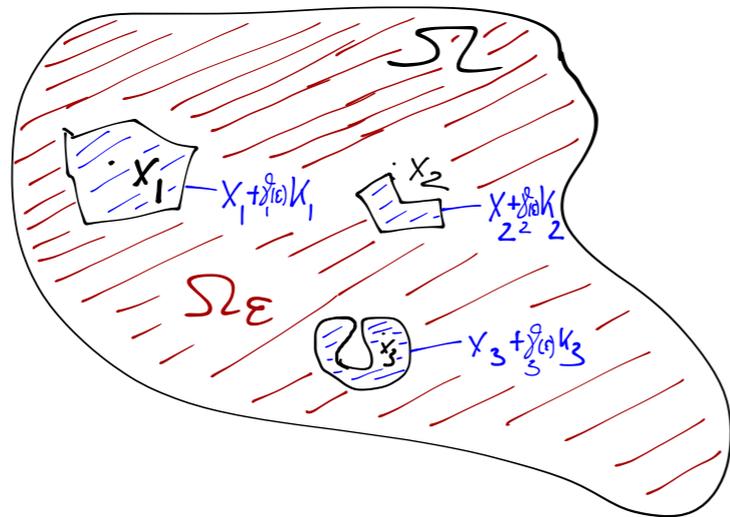
$$(2) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

Initial situation:

The aim is to prove that if  $f_\varepsilon \rightarrow f$  in  $L^p(\Omega)$  for some  $p > \frac{d}{2}$ , then the weak solutions  $u_\varepsilon$  of (1) converge uniformly to the unique weak solution  $u$  of (2)



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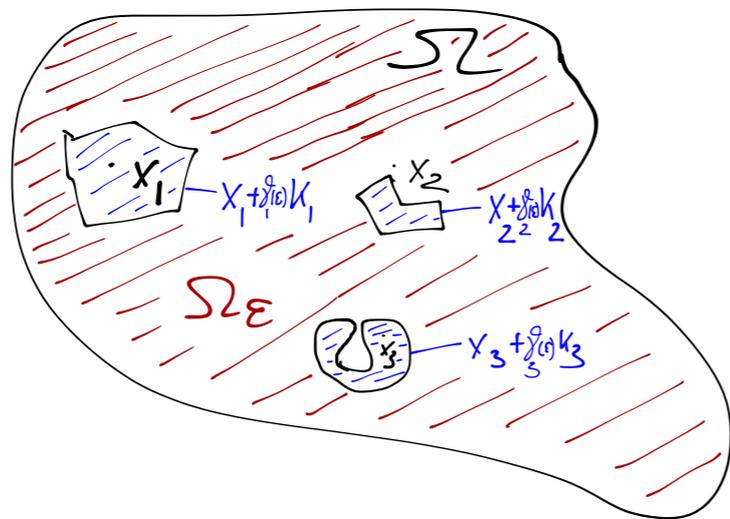
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Note: A similar result for Dirichlet problems cannot be true. ▽

## Remarks

▷  $L^2$ - or  $L^p$ -convergence of solutions to elliptic equations on domains with shrinking holes has been studied by many authors as, for instance,

- Daners '90
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- Daners '99, '08
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- ▷ In contrast to this, we have established  $L^\infty$ -convergence of solutions & we do not just treat Neumann bdy conditions, but also Robin bdy conditions where  $\beta_\varepsilon$  may be positive or negative.



# 1. Theorem (Dancer, Daves, #14)

Suppose the bdy-function  $\beta_\varepsilon: \partial H_\varepsilon \rightarrow \mathbb{R}$  satisfy *Assumption 1*.

If  $f_\varepsilon \rightarrow f$  in  $L^p(\Omega)$  for some  $p > \frac{d}{2}$ , then the weak solutions  $u_\varepsilon$  of

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converges uniformly to the unique solution  $u$  of

$$(2) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where "uniform convergence" means

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{x \in \Omega_\varepsilon} |u_\varepsilon(x) - u(x)| = 0.$$



For simplicity, we write the assumptions on  $\beta_\varepsilon$  only for  $\gamma_j(\varepsilon) = \varepsilon$ .

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(1)  $\exists \alpha \in (0, 1)$  s.t.  $\limsup_{\varepsilon \rightarrow 0^+} \varepsilon^\alpha \|\beta_\varepsilon\|_\infty < \infty$  if  $\beta_\varepsilon$  is sign changing or  $\beta_\varepsilon < 0$ ,



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# Uniform estimates on domains with small holes



## Uniform estimates on domains with small holes

- ▷ Solutions of elliptic bdy-value problems have a *global smoothing property* in Lebesgue spaces provided the underlying space of solutions  $H^1(\Omega_\varepsilon)$  admits a "Sobolev-type inequality":

$$\|\sigma\|_{L^{d^*}(\Omega_\varepsilon)} \leq C(\Omega_\varepsilon) \|\sigma\|_{H^1(\Omega_\varepsilon)} \quad \forall \sigma \in H^1(\Omega_\varepsilon),$$

where  $d^* = \frac{2d}{d-2}$  if  $d \geq 3$ , &  $d^* < \infty$  if  $d=2$  &

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Note:  $\partial\Omega_\varepsilon$  is Lipschitz! Once one has  $\textcircled{*}$   $\xRightarrow{\text{by Moser iteration method}}$   $L^p$ - $L^q$  or  $L^p$ - $L^\infty$ -estimates of the solutions.



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# Uniform estimates on domains with small holes

## Uniform estimates on domains with small holes

### 2. Theorem (Dancer, Daners, H., 14)

Suppose that **Assumption 1** is satisfied. Let  $p > \frac{d}{2}$  and  $f_\varepsilon \in L^p(\Sigma_\varepsilon)$  for all  $\varepsilon$ .

Then  $\exists \varepsilon_0 > 0$  &  $M > 0$  independent of  $\varepsilon \in (0, \varepsilon_0]$

such that the solution  $u_\varepsilon$  of (1) 
$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Sigma_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial \Sigma \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \partial H_\varepsilon \end{cases}$$

satisfies

$$\|u_\varepsilon\|_\infty \leq M \cdot \|f_\varepsilon\|_p \quad \text{for all } \varepsilon \in (0, \varepsilon_0]$$



## Consequences of uniform $L^p$ - $L^\infty$ -estimates

- ▷ The convergence of  $u_\varepsilon$  to  $u$  in  $L^p(\Omega)$  as  $\varepsilon \rightarrow 0+$ .



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- ▷ Uniform convergence away from the holes;

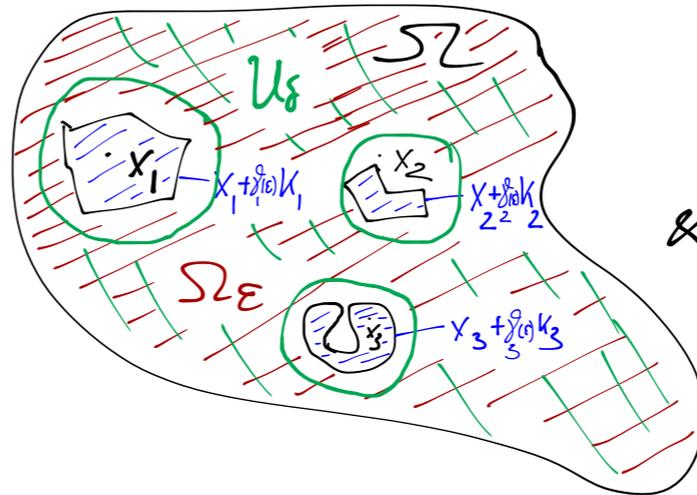


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- ▷ The convergence of  $u_\varepsilon$  to  $u$  in  $L^p(\Omega)$  as  $\varepsilon \rightarrow 0+$ .
- ▷ Uniform convergence away from the holes;

$\implies$  The proof of Theorem 1 reduces to showing that  $\forall \delta > 0$  we have

This means that the initial problem becomes a "local" problem.  $\left\{ \sup_{x \in \Omega_\varepsilon \setminus U_\delta} |u_\varepsilon(x) - u(x)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0+ \right.$



&  $\forall \delta > 0$  let

$$U_\delta := \Omega \setminus \bigcup_{j=1}^m B(x_j, \delta).$$

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- ▷ We may assume that  $u_\varepsilon$  &  $u$  are harmonic close to the holes.



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$\Rightarrow$  We need to deal with only one  $K_i$  at a time.  
By translation, we may assume that  $x_i = 0$ .



## Consequences of uniform $L^p$ - $L^\infty$ -estimates

$\Rightarrow$  We need to deal with only hole  $K_1$  at a time.  
By translation, we may assume that  $x_1 = 0$ .

$\curvearrowright$  The new problem which need to consider instead of problem (1) is:

$$\tilde{(1)} \quad \begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \tilde{\Omega}_\varepsilon := B(0,1) \setminus \varepsilon K_1 \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \varepsilon \partial K_1 \end{cases}$$



# Rescaling to a hole of fixed size



## Rescaling to a hole of fixed size

We set  $\vartheta_\varepsilon(x) := u_\varepsilon(\varepsilon \cdot x)$ ,  $\tilde{\beta}_\varepsilon(x) := \beta_\varepsilon(\varepsilon x)$ .



## Rescaling to a hole of fixed size

We set  $v_\varepsilon(x) := u_\varepsilon(\varepsilon \cdot x)$ ,  $\tilde{\beta}_\varepsilon(x) := \beta_\varepsilon(\varepsilon x)$ .

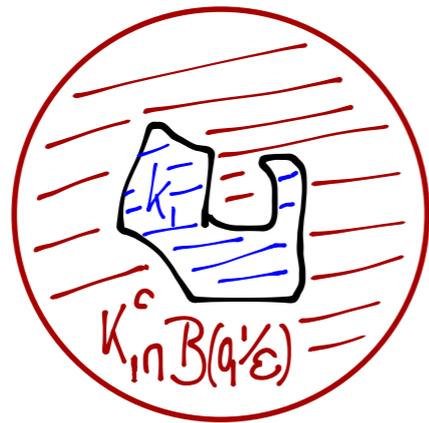
→  $v_\varepsilon$  is a weak solution of

$$(3) \quad \begin{cases} -\Delta v_\varepsilon = 0 & \text{in } \varepsilon^{-1} \tilde{\Omega}_\varepsilon = B(0, \frac{1}{\varepsilon}) \setminus K_1 \\ \frac{\partial v_\varepsilon}{\partial \nu} + \varepsilon \tilde{\beta}_\varepsilon v_\varepsilon = 0 & \text{on } \partial K_1 \end{cases}$$

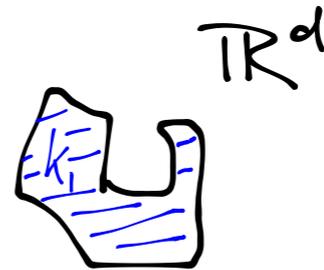
satisfying  $\|v_\varepsilon\|_\infty = \|u_\varepsilon\|_\infty \leq M \quad \forall \varepsilon \in (0, \bar{\varepsilon})$ .



Letting  $\varepsilon \rightarrow 0+$  in the rescaled problem



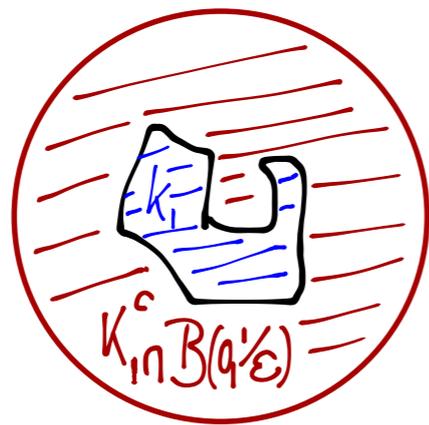
$\varepsilon \rightarrow 0+$   
→



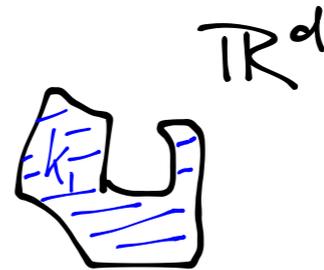
Exterior domain



Letting  $\varepsilon \rightarrow 0+$  in the rescaled problem



$\varepsilon \rightarrow 0+$   
 $\longrightarrow$



Exterior domain

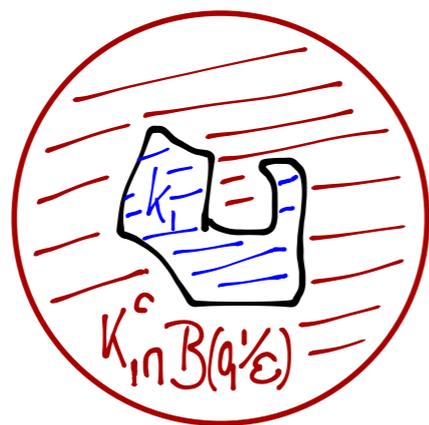
Since  $\|\vartheta_\varepsilon\|_\infty \leq M \forall \varepsilon > 0$  & by Assumption 1 on  $\beta_\varepsilon$ :

$\hookrightarrow (\vartheta_\varepsilon)$  is bdd in  $H^1_{loc}$ ,  $\hookrightarrow \vartheta_\varepsilon \rightarrow \vartheta$  in  $H^1_{loc}$   
 $\vartheta$  is a weak solution

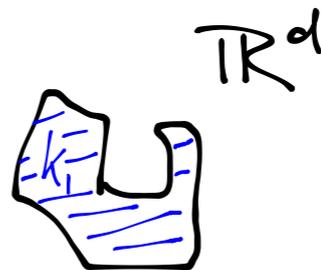
$$\text{of } \begin{cases} -\Delta \vartheta = 0 & \text{in } K_1^c \\ \frac{\partial \vartheta}{\partial \nu} = 0 & \text{on } \partial K_1 \end{cases}$$



Letting  $\varepsilon \rightarrow 0+$  in the rescaled problem



$\varepsilon \rightarrow 0+$   
 $\longrightarrow$



Exterior domain

Since  $\|v_\varepsilon\|_{L^\infty} \leq M \forall \varepsilon > 0$  & by Assumption 1 on  $\beta_\varepsilon$ :

$\hookrightarrow (v_\varepsilon)$  is bdd in  $H^1_{loc}$ ,  $\hookrightarrow v_\varepsilon \rightarrow v$  in  $H^1_{loc}$   
 $v$  is a weak solution

$$\text{of } \begin{cases} -\Delta v = 0 & \text{in } K_1^c \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial K_1 \end{cases}$$

Liouville fun ([Danos, Dakers, H.'13])

$\implies v \equiv v_0$  on  $K_1^c$  for some  $v_0 \in \mathbb{R}$ .



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At this point, we need to treat the case  $d \geq 3$   
and  $d=2$  separately.



Proof of Theorem 1 for  $d \geq 3$ :



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If  $v$  is a harmonic function on an annulus

$$A_{r_1, r_2} := \{x \in \mathbb{R}^d \mid r_1 < |x| < r_2\}$$

for some  $0 < r_1 < r_2$ ,

then  $\bar{v}(r) := \int_{\mathbb{S}_d} v(r, \theta) d\theta \quad \forall r \in (r_1, r_2)$

is a solution of  $\frac{d}{dr} \left( r^{d-1} \frac{dv}{dr} \right) = 0$  on  $(r_1, r_2)$ .



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$$\implies \bar{v}_\varepsilon \text{ satisfies } \frac{d}{dr} \left( r^{d-1} \frac{d\bar{v}}{dr} \right) = 0 \text{ on } (R_1, \frac{1}{\varepsilon})$$

where  $R_1 := \sup\{|y| \mid y \in K_1\} < 1$  <sup>by assumption</sup>

$\bar{u}_\varepsilon$  &  $\bar{u}$  satisfy this equation on  $(\varepsilon R_1, 1)$ .



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Since  $\{1, E_d\}$  is a basis of the space of radially symmetric harmonic functions on any annulus &  $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$

$$\begin{aligned} \implies \bar{v}_\varepsilon(r) &= a_\varepsilon + b_\varepsilon E_d(r) \\ \bar{u}_\varepsilon(r) &= a_\varepsilon + b_\varepsilon E_d(r/\varepsilon) = a_\varepsilon + b_\varepsilon \varepsilon^{d-2} E_d(r) \\ \bar{u}(r) &= u(0) \quad (\text{Mean value property}) \end{aligned}$$



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$$v_{\varepsilon_2'} \rightarrow v_0 \text{ in } \mathcal{C}(\overline{K_1^c \cap B(0, R)}) \quad \forall R > 1,$$

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From (1) & since  $\{1, E_d\}$  is linear independent

$$\bar{v}_{\varepsilon}(r) = a_{\varepsilon} + b_{\varepsilon} E_d(r) \Rightarrow b_{\varepsilon} \rightarrow 0 \quad \& \quad a_{\varepsilon} \rightarrow v_0$$



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$$\& \text{ so also } \varepsilon^{d-2} b_{\varepsilon_2'} \rightarrow 0.$$

$$\text{Using } \bar{u}_\varepsilon(\tau) = a_\varepsilon + b_\varepsilon \varepsilon^{d-2} E_d(\tau) \stackrel{(2)}{\Rightarrow} \underline{\underline{u(0) = v_0}}$$

□



Proof of Theorem 1 for  $d=2$  :



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Since  $E_2(r) = \log r$ , we have

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But since  $\log \varepsilon \rightarrow -\infty \not\Rightarrow b_\varepsilon \cdot \log \varepsilon \rightarrow 0$ .



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We can complete the proof if we assume that

$$\mathbb{R}^2 \setminus K_1 \cong \mathbb{R}^2 \setminus B(0,1)$$

& by using the boundary cond. on  $\partial \varepsilon K_1$ .  $\square$



Thank You!

