Example: Let $M = N = F[x]$, regarded as vector spaces over a field F .

What is $M \otimes_F N$?

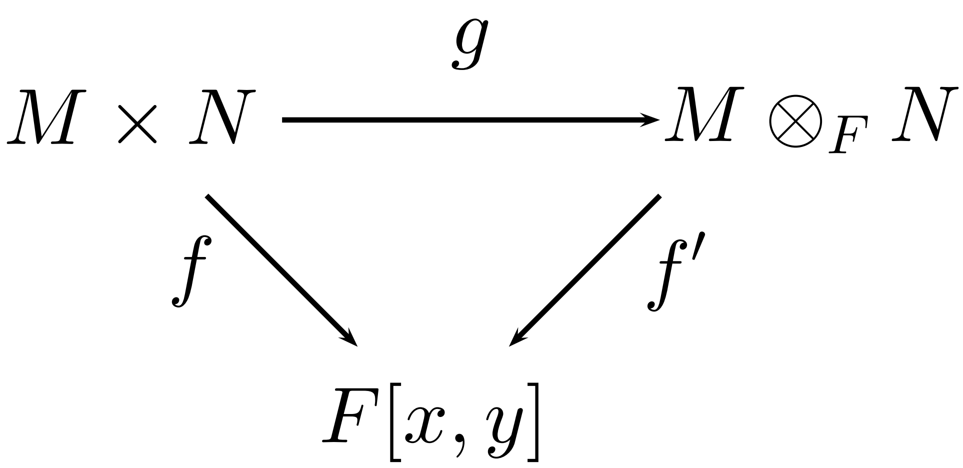
To answer this: Consider the bilinear mapping

$$f \ : \ M \times N \ \longrightarrow \ F[x,y]$$

defined by

$$\big(\, p(x)\,,\; q(x)\,\big) \ \mapsto \ p(x)\, q(y) \ .$$

The Theorem guarantees the existence of a unique module homomorphism f' such that the following diagram commutes:



Note that

$$M = N = \langle\, x^i \mid i \geq 0 \,\rangle$$

so that

$$M \otimes N \;=\; \langle\, x^i \otimes x^j \;\mid\; i, j \geq 0 \,\rangle \,.$$

Note also that, for each $i, j \geq 0$,

$$\begin{aligned} f'(x^i \otimes x^j) \;&=\; f'(g(x^i, x^j)) \\ &=\; f(x^i, x^j) \;=\; x^i y^j \,. \end{aligned}$$

We wish to show that f' is an isomorphism.

Define $h : F[x,y] \longrightarrow M \otimes N$ by

$$h\Big(\sum \lambda_{m,n}\, x^m\, y^n \Big) \;=\; \sum \lambda_{m,n} \left(x^m \otimes x^n \right) .$$

Easy to check h is linear, and further that, for $i, j \geq 0$

$$(f' \circ h)(x^i y^j) \;=\; f'(x^i \otimes x^j) \;=\; x^i y^j$$

and

$$(h \circ f')(x^i \otimes x^j) \;=\; h(x^i y^j) \;=\; x^i \otimes x^j .$$

Thus, these mapping fix the generators of $F[x,y]$ and $M \otimes N$ respectively, so

$$f' \circ h \;=\; \mathrm{id}_{F[x,y]} \qquad \text{and} \qquad h \circ f' \;=\; \mathrm{id}_{M \otimes N} \,,$$

from which it follows immediately that f' and h are mutually inverse isomorphisms. This proves

$$\boxed{F[x] \otimes_F F[x] \;\cong\; F[x,y] \,.}$$

What about $F[x] \otimes_{F[x]} F[x]$

where $F[x]$ (being a ring also) is regarded as a module over itself?

Exercise: Prove that, for any A-module M,

$$A \otimes_A M \cong M ,$$

extending the following mapping on generators:

$$a \otimes x \mapsto a\, x \qquad (x \in M , \; a \in A) .$$

As an illustration of technique we will prove the following result:

Proposition: Let M , N be A-modules. Then there is a unique isomorphism

$$M \otimes_A N \cong N \otimes_A M$$

extending the following mapping on generators:

$$x \otimes y \mapsto y \otimes x \qquad (y \in N , x \in M) .$$

Proof: Define $f : M \times N \longrightarrow N \otimes M$ by

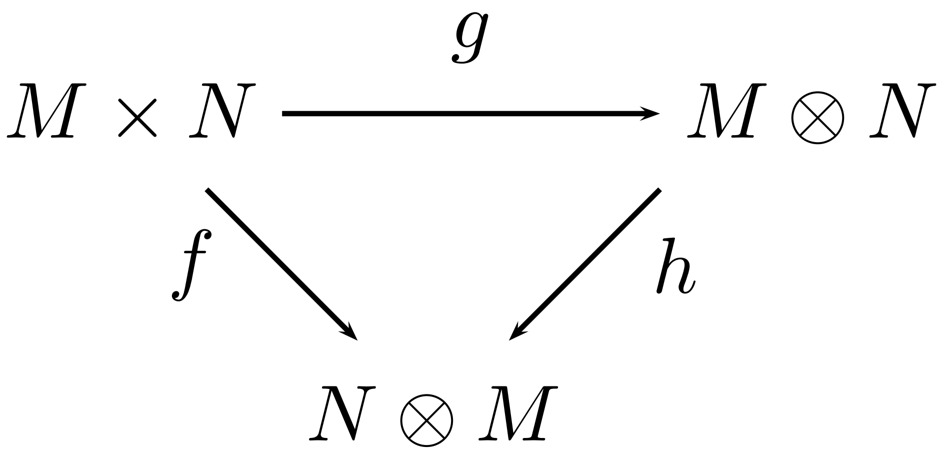
$$f(x, y) = y \otimes x \qquad (x \in M, y \in N) .$$

Then, by properties of tensors,

$$\begin{aligned} f(ax_1 + bx_2, y) &= y \otimes (ax_1 + bx_2) \\ &= (y \otimes ax_1) + (y \otimes bx_2) \\ &= a(y \otimes x_1) + b(y \otimes x_2) \\ &= af(x_1, y) + bf(x_2, y) . \end{aligned}$$

Similarly f is linear in the second variable.

Thus f is bilinear, so there is a unique module homomorphism h such that the following diagram commutes:



Hence, for $x \in M$, $y \in N$,

$$h(x \otimes y) \;=\; h(g(x, y)) \;=\; f(x, y) \;=\; y \otimes x \; .$$

Similarly, there exists a module homomorphism k : $N \otimes M \rightarrow M \otimes N$ such that

$$k(y \otimes x) = x \otimes y \qquad (x \in M)(y \in N) .$$

whence

$$(k \circ h)(x \otimes y) = k(y \otimes x) = x \otimes y .$$

Hence $k \circ h = \mathrm{id}_{M \otimes N}$ (since it fixes generators).

Similarly $h \circ k = \mathrm{id}_{N \otimes M}$, so h and k are mutually inverse isomorphisms. Further h is unique because it is defined on generators.

Despite the ambiguity of tensors of elements, we have

Corollary to the proof of the main Theorem:

Suppose $\sum_{i=1}^{n} (x_i \otimes y_i) \;=\; 0$ in $M \otimes N$. Then

$$\sum_{i=1}^{n} (x_i \otimes y_i) \;=\; 0 \qquad \text{in} \qquad M_0 \otimes N_0 \,,$$

for some finitely generated submodules M_0 of M and N_0 of N .

Proof: To say

$$\sum (x_i \otimes y_i) \;=\; 0 \qquad \text{in} \qquad M \otimes N\ ,$$

means

$$\sum (x_i\, ,\, y_i) \;\in\; D$$

where $M \otimes N \;=\; C/D$ for C , D as in the proof of the main Theorem.

Hence $\sum (x_i, y_i)$ is a finite linear combination of generators of D .

Let M_0 be the submodule of M generated by all (finitely many) elements of M appearing in first coordinates of ordered pairs used in these generators.

Define N_0 analogously using second coordinates.

Both M_0 and N_0 are finitely generated.

Then $M_0 \otimes N_0 = C_0/D_0$ for appropriate C_0 and D_0 .

But $\sum (x_i, y_i) \in D \cap C_0 = D_0$, so

$$\sum (x_i, y_i) \in D_0 \qquad \text{in} \qquad M_0 \otimes N_0 \; .$$

Tensoring homomorphisms:

Let $f : M \to M'$, $g : N \to N'$ be A-module homomorphisms. We wish to use these to construct a homomorphism

$$f \otimes g \; : \; M \otimes N \; \to \; M' \otimes N' \, .$$

Define

$$h \; : \; M \times N \; \to M' \otimes N'$$

by

$$h(x, y) \;\; = \;\; f(x) \otimes g(y) \; .$$

Then h is linear in the first variable:

$$
\begin{aligned}
h(ax_1 + bx_2, y) &= f(ax_1 + bx_2) \otimes g(y) \\
&= \big(af(x_1) + bf(x_2)\big) \otimes f(y) \\
&= a\big(f(x_1) \otimes g(y)\big) \; + \; b\big(f(x_2) \otimes g(y)\big) \\
&= a\,h(x_1, y) \; + \; b\,h(x_2, y) \; .
\end{aligned}
$$

Similarly in the second variable, so $\boxed{h \text{ is bilinear.}}$

Hence there is a unique homomorphism $f \otimes g$ which makes the following diagram commute:

$$\begin{array}{ccc} M \times N & \xrightarrow{} & M \otimes N \\ & {\scriptstyle h}\searrow \qquad \swarrow{\scriptstyle f\otimes g} & \\ & M' \otimes N' & \end{array}$$

so that

$$\boxed{\big(f \otimes g\big)(x \otimes y) \ = \ f(x) \otimes g(y)\ .}$$

Example: Let F be a field and fix $\alpha, \beta \in F$. Define linear transformations $f, g : F[x] \longrightarrow F$ by

$$f\big(p(x)\big) = p(\alpha) \qquad \text{and} \qquad g\big(p(x)\big) = p(\beta) .$$

Then

$$f \otimes g \; : \; F[x] \otimes_F F[x] \; \longrightarrow \; F \otimes_F F$$

where

$$p_1(x) \otimes p_2(x) \; \longmapsto \; p_1(\alpha) \otimes p_2(\beta) .$$

Recall that $\phi : F[x, y] \longrightarrow F[x] \otimes_F F[x]$ extending

$$x^i y^j \mapsto x^i \otimes x^j$$

and $\psi : F \otimes_F F \longrightarrow F$ extending

$$a \otimes b \mapsto a\, b$$

are isomorphisms (the latter being a special case of an exercise).

Hence we get the following commutative square for some h :

$$
\begin{array}{ccc}
F[x,y] & \xrightarrow{\quad h \quad} & F \\
\phi \big\downarrow & & \big\uparrow \psi \\
F[x] \otimes_F F[x] & \xrightarrow[\quad f \otimes g \quad]{} & F \otimes_F F
\end{array}
$$

so that

$$
\begin{aligned}
h\Big(\sum \lambda_{i,j}\, x^i\, y^j \Big) &= \psi\Big(f \otimes g \, \Big(\phi\Big(\sum \lambda_{i,j}\, x^i\, y^j \Big)\Big)\Big) \\
&= \psi\Big(f \otimes g \, \Big(\sum \lambda_{i,j}\, x^i \otimes x^j \Big)\Big) \\
= \psi\Big(\sum \lambda_{i,j}\, \alpha^i \otimes \beta^j \Big) &= \sum \lambda_{i,j}\, \alpha^i\, \beta^j \, .
\end{aligned}
$$

Tensoring behaves well with respect to composition of homomorphisms:

If
$$M \overset{f}{\longrightarrow} M' \overset{f'}{\longrightarrow} M''$$
and
$$N \overset{g}{\longrightarrow} N' \overset{g'}{\longrightarrow} N''$$

are A-module homomorphisms then

$$\boxed{\big(f' \circ f\big) \otimes \big(g' \circ g\big) \;=\; \big(f' \otimes g'\big) \circ \big(f \otimes g\big)}$$

because they agree on generators:

$$\Big[\big(f' \circ f\big) \otimes \big(g' \circ g\big)\Big](x \otimes y) = \Big[\big(f' \circ f\big)(x)\Big] \otimes \Big[\big(g' \circ g\big)\Big](y)\Big]$$

$$= \quad f'(f(x)) \;\otimes\; g'(g(y)) \quad = \quad \big(f' \otimes g'\big)\big(f(x) \;\otimes g(y)\big)$$

$$= \quad \big(f' \otimes g'\big)\Big[\big(f \otimes g\big)(x \otimes y)\Big]$$

$$= \quad \Big[\big(f' \otimes g'\big) \circ \Big[\big(f \otimes g\big)\Big](x \otimes y)\ .$$