(A)

Commutative Algebra — Lecture 37 — Mon 1/6/09

Composition series & Jordan-Holder

Throughout let M be a nontrivial A-module.

Call M simple if $M \neq \{0\}$ and M has no nontrivial proper submodules,

i.e. $\{0\}$ is a maximal submodule,

i.e. lattice of submodules of M is

$$\begin{tikzcd} M \arrow[d, dash] \\ \{0\} \end{tikzcd}$$

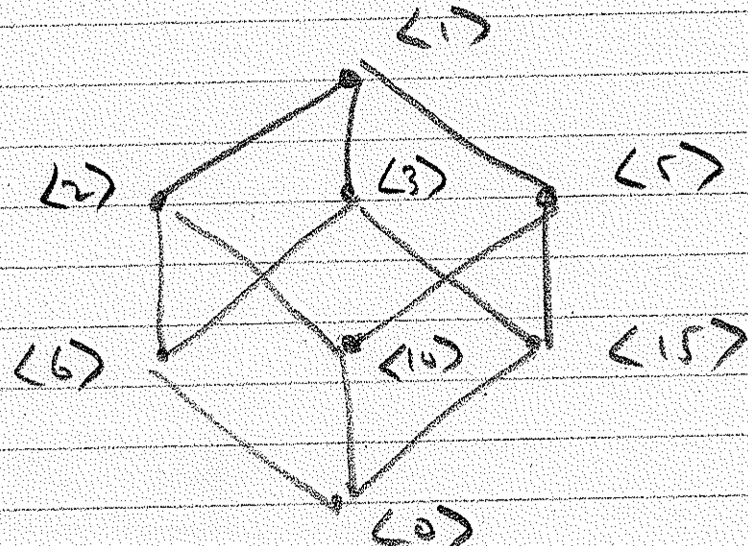
A series for M is a sequence of submodules

$$\mathcal{S}: \quad \{0\} = M_0 \leq M_1 \leq \dots \leq M_{n-1} \leq M_n = M$$

Call $\mathcal{S}$ a composition series if each factor $\underbrace{M_{i+1}/M_i}_{\text{called a composition factor}}$ is simple $(i = 0, \dots, n-1)$.

Call n the length of $\mathcal{S}$.

e.g. $\mathbb{Z}_{30} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$



$\langle 0\rangle \le \langle 6\rangle \le \langle 2\rangle \le \langle 1\rangle = \mathbb{Z}_{30}$

with composition factors

$\langle 6\rangle / \langle 0\rangle \cong \mathbb{Z}_5$, $\langle 2\rangle / \langle 6\rangle \cong \mathbb{Z}_3$,

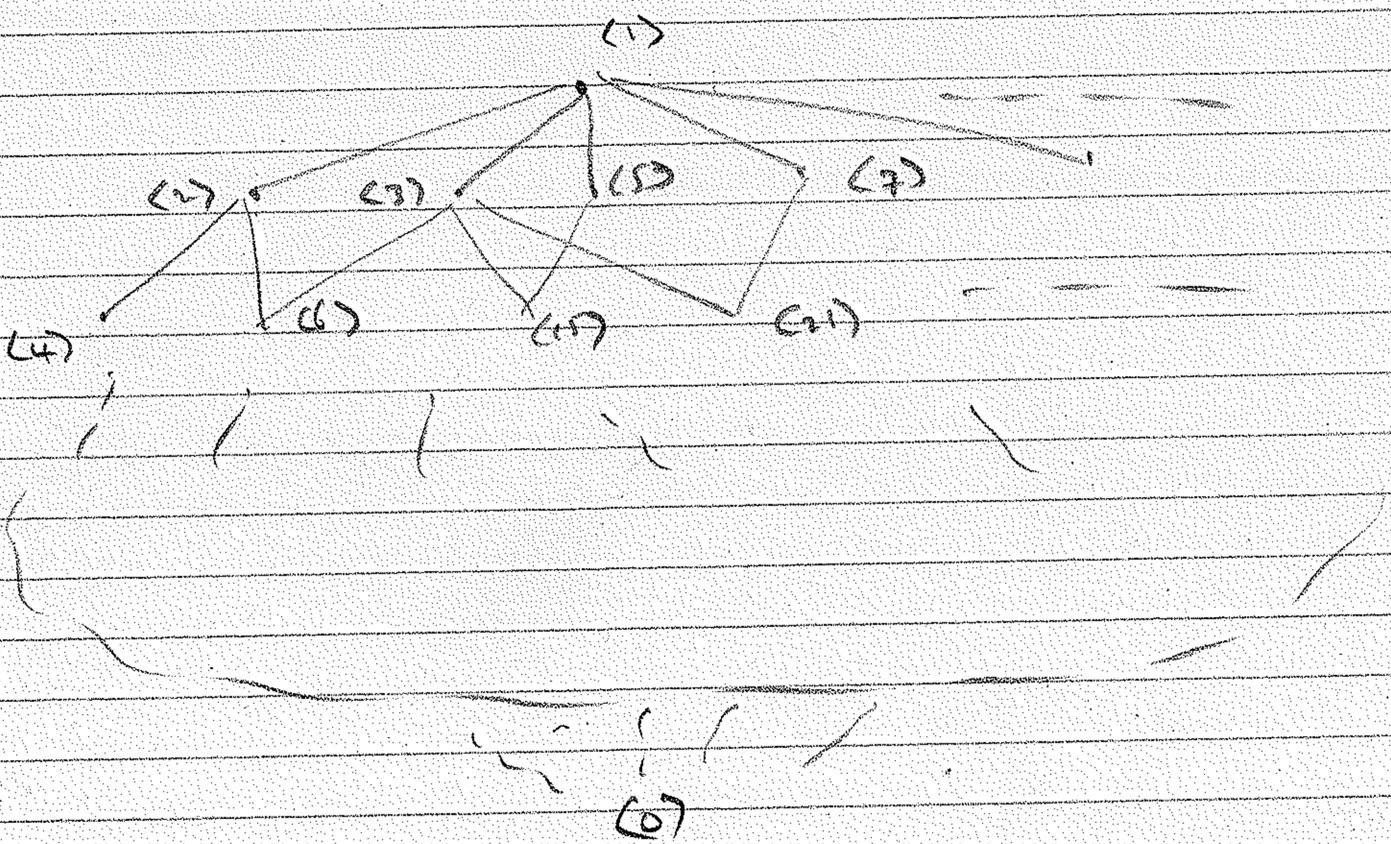
$\langle 1\rangle / \langle 2\rangle \cong \mathbb{Z}_2$.

<u>Jordan-Holder Theorem</u>: If M has a composition series of length n then all composition series of M have length n, and $\exists$ one-one correspondence of isomorphic composition factors.

c

e.g. In $\mathbb{Z}_{30}$ all 6 composition series have length 3 and the composition factors are one copy of each of $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$.

e.g. $\mathbb{Z}$ has no composition series



but every proper interval avoiding $\langle 0 \rangle$ does have a composition series ($\cong$ lattice of submodules of some $\mathbb{Z}_n$)

Lemma: If M has a composition series then so does every submodule of M.

Proof: Suppose $N \leq M$ and M has a composition series

$$S : \{0\} = M_0 \leq M_1 \leq \ldots \leq M_{k-1} \leq M_k = M.$$

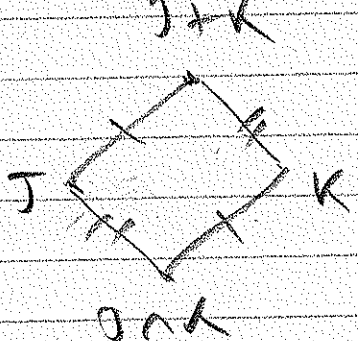
Form the following series for N:

$$S' : \{0\} = M_0 \cap N \leq M_1 \cap N \leq \ldots \leq M_k \cap N = N$$

But

$$\frac{M_{i+1} \cap N}{M_i \cap N} = \frac{M_{i+1} \cap N}{M_i \cap (M_{i+1} \cap N)}$$

since $M_i \leq M_{i+1}$



$$\cong \frac{M_i + (M_{i+1} \cap N)}{M_i}$$

$$\leq M_{i+1} / M_i,$$

which is simple.

Hence $\dfrac{M_{i+1} \cap N}{M_i \cap N}$ is trivial or simple,

so deleting duplicates from $\mathcal{S}'$ yields a composition series for N. $\square$

Proof of Jordan-Holder:

Suppose

$$\mathcal{S} : \{0\} = M_0 \leq M_1 \leq \ldots \leq M_{n-1} \leq M_n = M$$

is a composition series for M of length n.

If $n=1$ then $M = M_1$ is simple, so $\mathcal{S}$ is unique, which starts an induction.

Suppose $n > 1$ and J-H holds for modules with composition series of length $< n$.

Let

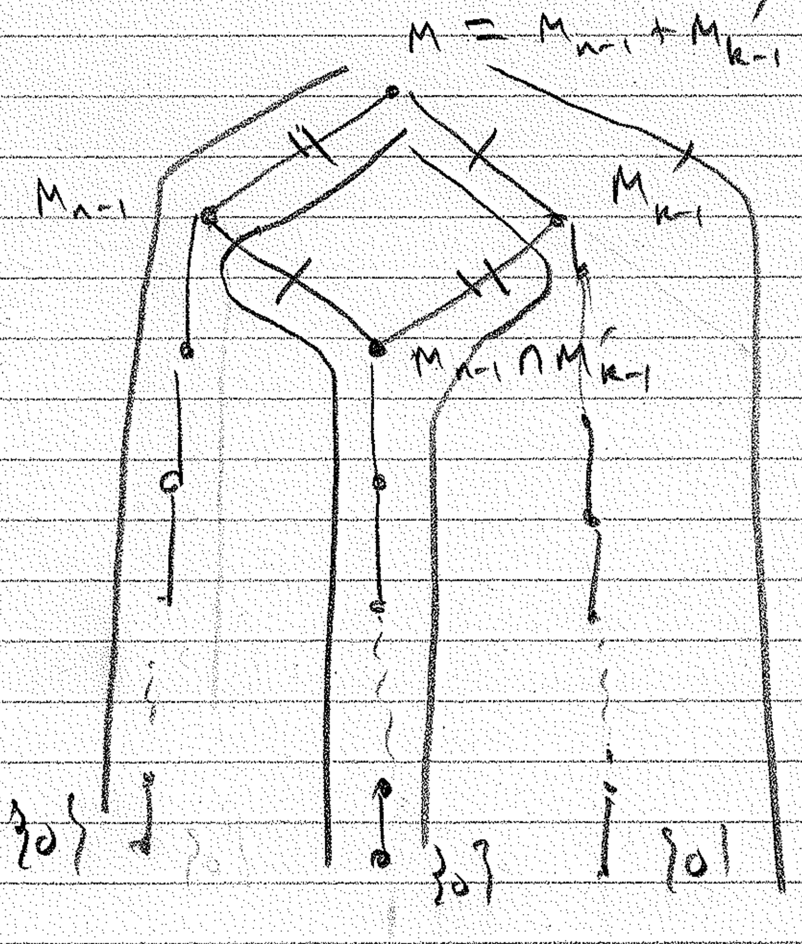
$$\mathcal{S}' : \{0\} = M_0' \leq M_1' \leq \ldots \leq M_{k-1}' \leq M_k' = M$$

be another composition series of M.

F

<u>Case (i)</u> : If $M_{n-1} = M'_{k-1}$ then, by ind. hyp., $n-1 = k-1$, so $n = k$, and composition factors for M_{n-1} match up for S and S', plus $M/M_{n-1} = M/M'_{k-1}$, so done. ✓

<u>Case (ii)</u> : If $M_{n-1} \neq M'_{k-1}$ then apply isomorphism theorem, ind. hyp., and Lemma to $M_{n-1} \cap M'_{k-1}$:



and everything matches up, and done ✓ □