Euler and Pontryagin classes

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Reference:

A. Hatcher. "Vector bundles and K-theory", §§3.1 – 3.2. Available online.

§1. Reminder on universal bundles

- Recall: We have Grassmann manifolds

$$G_n(\mathbb{R}^k) = \{n\text{-dim subspaces of } \mathbb{R}^k\}$$
$$\subseteq G_n(\mathbb{R}^{k+1})$$
$$\subseteq \cdots \subseteq G_n(\mathbb{R}^\infty) = \bigcup_{k \geq n} G_n(\mathbb{R}^k)$$

and canonical rank n bundles

$$E_n(\mathbb{R}^k) = \{(\ell, v) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k : v \in \ell\} \longrightarrow G_n(\mathbb{R}^k).$$
$$\subseteq E_n(\mathbb{R}^{k+1})$$
$$\subseteq \cdots \subseteq E_n(\mathbb{R}^\infty) = \bigcup_{k \geq n} E_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^\infty)$$

- Let $G_n = G_n(\mathbb{R}^\infty)$ = classifying space of rank n bundles,

$E_n = E_n(\mathbb{R}^\infty)$ – universal bundle.

- <u>Theorem</u>: For paracompact X, the map
$$[X, G_n] \longrightarrow \mathrm{Vect}^n(X), \ [f] \mapsto f^*(E_n)$$
is a bijection.

- <u>theoretical application</u>: To prove every rank n
$$E \longrightarrow X \text{ (paracompact)}$$
admits an inner product, we can:
  (1) Define an inner product on E_n by restricting from $\mathbb{R}^\infty$ to each n-plane.
  (2) Express $E \cong f^*(E_n)$.
  (3) Show there is an induced inner product on E.

- <u>Oriented versions</u>: $\widetilde{G}_n(\mathbb{R}^k)$, $\widetilde{E}_n(\mathbb{R}^k)$, $\widetilde{G}_n$, and $\widetilde{E}_n$ all defined with oriented n-planes replacing standard n-planes. Then have a version of the theorem for oriented bundles.

- **Goal:** Compute $H^*(G_n, \mathbb{Z})$ and $H^*(\tilde{G}_n, \mathbb{Z})$.

- **Observe:** A characteristic class can be defined as a function associating a v.b.

$$E \longrightarrow B \quad (\text{rank } n)$$

to a class $x(E) \in H^k(B; G)$ for some group G and some k, such that

$$x(f^*(E)) = f^*(x(E)). \qquad (*)$$

Using $(*)$ and the universal property of E_n, we see that

$$\{\text{char classes}\} \xrightarrow{\cong} H^k(G_n; G)$$
$$x \longmapsto x(E_n).$$

- **Already know:**

$$H^*(G_n; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[w_1, \ldots, w_n], \quad \text{Stiefel-Whitney classes}$$

$$H^*(G_n(\mathbb{C}); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n] \quad \text{Chern classes.}$$

§2. Euler classes

R = ring, $p: E \to B$ oriented, real, of rank n.

Thom: $\exists\, c \in H^n(E, E - s_0(B); R)$ such that $c|_{(F, F-0)} \in H^n(F, F-0; R)$ is the class induced by the orientation of a fibre F. Moreover, c is a free generator for $H^*(E, E - s_0(B); R)$ as a graded right $H^*(E; R)$-module.

- Application: Euler class $e(E)$ defined by

$$H^n(D(E), S(E); \mathbb{Z}) \to H^n(D(E); \mathbb{Z}) \to H^n(B; \mathbb{Z})$$
$$c \mapsto e(E).$$

- Basic properties: Let $E \to B$ be an oriented vector bundle.

(a) There is an induced orientation of a pullback $f^*(E) \to A$

$$\underset{A}{e}(f^*(E)) = f^*(\underset{B}{e}(E)).$$

(b) If $E' \to B$ is oriented, then $\exists$ orientation of $E \oplus E'$ s.t.

$$e(E \oplus E') = e(E) \smile e(E').$$

(c) $e(E) = -e(E)$ if E has odd rank.

(d) $e(E) = 0$ if E admits a non-vanishing section.

(e) $e(E) \mapsto w_n(E)$ under $H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}/2\mathbb{Z})$

and $e(E) = c_n(E) \in H^{2n}(B;\mathbb{Z})$, if E complex of dim n.

Proof: (a)
$$\begin{tikzcd} f^*(E) \arrow[r, "\bar{f}"] \arrow[d] & E \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$
Cartesian square

Here $\tilde{f}$ is an iso. in each fibre, so

$$\tilde{f}^*(c(E)) = c(f^*(E)).$$

$$\xrightarrow[\text{res. to zero section}]{\text{relative to absolute}} \quad f^*(e(E)) = e(f^*(E))$$

(b) Write

$$\begin{tikzcd} & E \oplus E' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E & & E' \end{tikzcd}$$

Then use that $p_1^*(c(E)) \smile p_2^*(c(E'))$ is a Thom class for $E_1 \oplus E_2$.

(c) Thom isomorphism:

$$H^n(B;\mathbb{Z}) \longrightarrow H^{2n}(D(E), S(E);\mathbb{Z}),$$

$$e(E) \longmapsto c \smile c.$$

Then recall $c \smile c = -c \smile c$, n odd.

(d) Say $s : B \to E$ everywhere non-zero.

$$\xrightarrow[\text{normalise}]{} \quad s : B \to S(E).$$

Now note:

$$\begin{tikzcd} D(E) \arrow[r] \arrow[rrr, bend right, "\mathrm{id}"', "\wr"] & B \arrow[r, "s"] & S(E) \arrow[r, "i"] & D(E) \end{tikzcd}$$

so i^* is injective. But

$$H^n(D(E), S(E); \mathbb{Z}) \xrightarrow{j^*} H^n(D(E); \mathbb{Z}) \xrightarrow{i^*} H^n(S(E); \mathbb{Z})$$

exact $\Rightarrow j^* = 0 \Rightarrow e(E) = 0$.

(e) Omitted.

- Example: Consider $TS^n \longrightarrow S^n$.

  Prop. $\Rightarrow e(TS^n) = 0$ for n odd.

  But for n even,

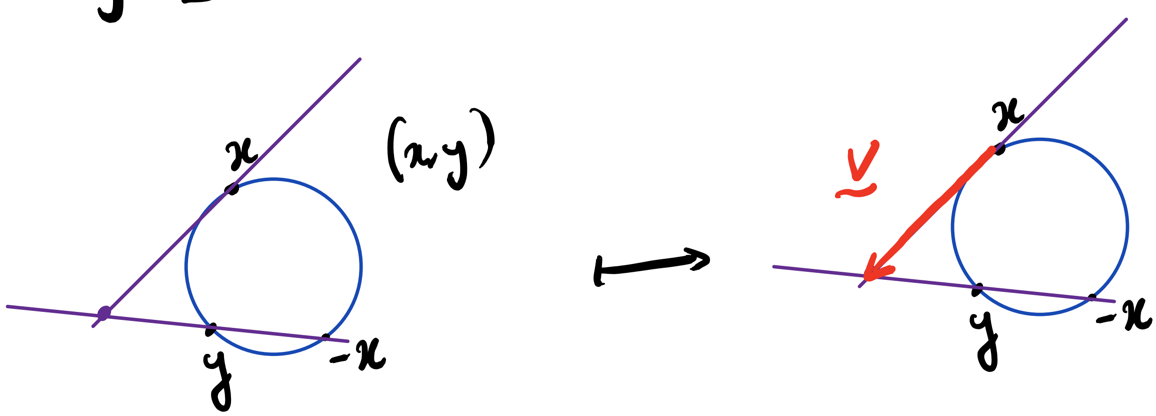
  $$e(TS^n) = 2 \cdot (\text{gen. of } H^n(S^n; \mathbb{Z})).$$

Proof: ETP $c^2 = c \smile c$ is twice a generator of $H^{2n}(E,E')$, where

$$E' = E - s_0(S^n) \subseteq E = TS^n.$$

Set $A = \{(x,-x) \in S^n \times S^n\}$ and consider the well-known homeomorphism

$$f : S^n \times S^n - A \xrightarrow{\simeq} E,$$ as depicted:



Note $\Delta = \{(x,x)\} \simeq E'$ under f, so that

$$\begin{aligned} H^*(E,E') &\overset{\text{excision}}{\cong} H^*(S^n \times S^n, S^n \times S^n - \Delta) \\ &\cong H^*(S^n \times S^n, A) \qquad \text{deformation retracts} \\ &\cong H^*(S^n \times S^n, \Delta). \end{aligned}$$

But the LES of $(S^n \times S^n, \Delta)$ yields

$$0 \to H^n(S^n \times S^n, \Delta) \to H^n(S^n \times S^n) \xrightarrow{r} H^n(\Delta) \to 0$$

$$H^n(S^n \times S^n) \cong \langle \alpha, \beta \rangle = \langle p_1^*(\gamma), p_2^*(\gamma) \rangle$$

where $S^n \times S^n \underset{p_2}{\overset{p_1}{\rightrightarrows}} S^n$ and $H^n(S^n) = \langle \gamma \rangle$.

Note $r(\alpha) = r(\beta)$, since $p_1|_\Delta = p_2|_\Delta$, so

$$\ker r = H^n(S^n \times S^n, \Delta) = \langle \alpha - \beta \rangle$$

In other words, $\alpha - \beta = \pm c$ so

$$c^2 = (\alpha - \beta)^2 = -\alpha\beta - \beta\alpha = -2\alpha\beta$$

for n even. $\square$

- <u>Generalisation:</u> If M is a smooth n-manifold, then

$$e(TM) = \chi(M) \cdot (\text{generator of } H^n(M)),$$

where $\chi(M)$ is the Euler characteristic of M. This explains the name.

§3. Pontryagin classes

- If $E \to B$ is an $\mathbb{R}$-vector bundle of rank n its complexification is the $\mathbb{C}$-vector bundle

$$E^{\mathbb{C}} = E \oplus E \longrightarrow B$$

where $i \in \mathbb{C}$ acts in fibres $\mathbb{R}^n \oplus \mathbb{R}^n$ by

$$(x, y) \longmapsto (-y, x).$$

- The Pontryagin class $p_i(E)$ is then defined to be

$$p_i(E) = (-1)^i c_{2i}(E^{\mathbb{C}}) \in H^{4i}(B; \mathbb{Z}).$$

- Prop: (a) Denote by γ_j the coefficient hom. $H^j(B; \mathbb{Z}) \longrightarrow H^j(B; \mathbb{Z}/2\mathbb{Z})$ Then γ_{4i}: $p_i(E) \longmapsto w_{2i}(E)^2$.

(b) If E is an orientable rank-$2n$ bundle, then $p_n(E) = e(E)^2$.

Proof: (a) It is an important fact that

$$\gamma_{2i}\left(c_i(F)\right) = w_{2i}(F_{\mathbb{R}}),$$

where F is a rank-n $\mathbb{C}$-vector bundle. Thus

$$\gamma_{4i}\left(c_{2i}(E^{\mathbb{C}})\right) = w_{4i}(E \oplus E).$$

But
$$\begin{aligned} w(E \oplus E) &= w(E)^2 \\ &= \left(1 + w_1(E) + w_2(E) + \cdots\right)^2 \\ &= 1 + w_1(E)^2 + w_2(E)^2 + \cdots \end{aligned}$$

Since we work mod 2 $\Rightarrow$ claim.

(b) Let $v_1, \ldots, v_{2n}$ an ordered basis for a fibre of E agreeing with its orientation.

$E^{\mathbb{C}}$ oriented by $v_1, iv_1, \ldots, v_{2n}, iv_{2n}$

$E \oplus E$ oriented by $v_1, \ldots, v_{2n}, iv_1, \ldots, iv_{2n}$

These agree after $n(2n-1)$ transpositions, where

$$n(2n-1) \equiv n \pmod 2,$$

so that
$$\begin{aligned} p_n(E) &= (-1)^n c_{2n}(E^{\mathbb{C}}) \\ &= (-1)^n e(E^{\mathbb{C}}) \quad \text{[recall §2]} \\ &= e(E \oplus E) = e(E)^2. \end{aligned}$$

- Pontryagin classes allow us to express the cohomology of G_n and $\tilde{G}_n$, up to torsion. Let

$$p_i = p_i(E_n), \quad \tilde{p}_i = p_i(\tilde{E}_n), \quad e = e(\tilde{E}_n).$$

- <u>Theorem:</u> (a) All torsion in $H^*(G_n, \mathbb{Z})$ and $H^*(\tilde{G}_n, \mathbb{Z})$ consists of elements of order 2.

  (b) We have

$$H^*(G_n, \mathbb{Z})/\text{torsion} \cong \mathbb{Z}[p_1, \dots, p_k],$$
$$n = 2k \text{ or } 2k+1;$$

$$H^*(\tilde{G}_n, \mathbb{Z})/\text{torsion} \cong \begin{cases} \mathbb{Z}[\tilde{p}_1, \dots, \tilde{p}_k], & n = 2k+1 \\ \mathbb{Z}[\tilde{p}_1, \dots, \tilde{p}_k, e], & n = 2k. \end{cases}$$