Linear response for macroscopic observables in high-dimensional systems

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Joint work with Georg Gottwald

Linear response theory

Consider a smooth family of deterministic dynamical systems $x_n = T^t(x_{n-1})$, which are mixing with physical invariant measures μ^t.:

$$\mathbb{E}^t[\Phi] := \int \Phi(x)\,\mathrm{d}\mu^t(x)$$

Linear response theory (LRT): *What is* $\frac{\mathrm{d}}{\mathrm{d}t}\mu^t\mathbb{E}^t[\Phi]$*?*
(e.g. for Taylor approximations)

. . . supposing $\mathbb{E}^t[\Phi]$ is differentiable

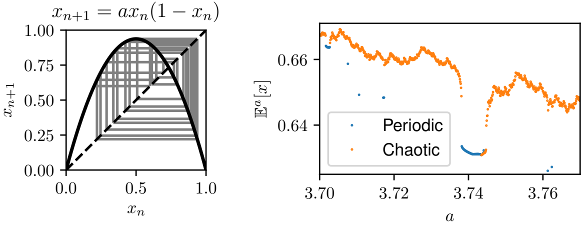
LRT in theory

Analytically, we know LRT works in

- Statistical mechanics (the original): Kubo '66
- Stochastic systems: Hänggi '78, Hairer & Majda '10 (range of Taylor series validity shrinks as noise $\to 0$)
- Hyperbolic deterministic systems: Ruelle '97-8

LRT in theory

Fails in simplest non-hyperbolic case: logistic map has Hölder $< 1/2$ response, even restricted to "nice" chaotic parameters (Baladi and others '08, '10, '14, '15)



Conjecture: "large scales of high-dimensional systems evolve hyperbolically, i.e. have LRT"

LRT in practice

LRT has been applied to plenty of systems:

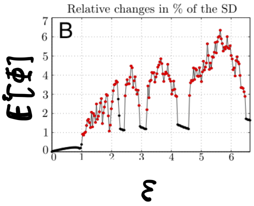
- Toy models: Majda et al '07, '10, Lucarini & Sarno '11
- Barotropic models: Bell '80, Gritsun & Dymnikov '99, Abramov & Majda '09
- Quasi-geostrophic models: Dymnikov & Gritsun '01
- Atmospheric models: North et al '04, Cionni *et al.* '04, work of Gritsun and others '02, '07, '10, Ring & Plumb '08
- Coupled climate models: Langen & Alexeev '05, Kirk & Davidoff '09, Fuchs et al '14, Ragone *et al.* '15, Lembo et al '20

Often it works pretty well!

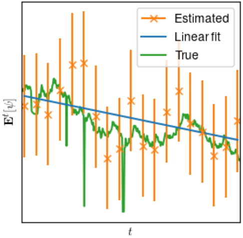
LRT in practice

However:

- Rough responses are known in atmospheric and ocean dynamics (e.g. Chekroun et al. '14)
- A lot of data is needed to see failure of linear respnose: maybe the picture is worse than thought (Gottwald, W. & Wouters '16)



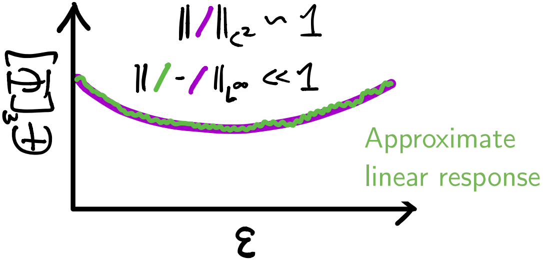
Chekroun et al., 2014



Question

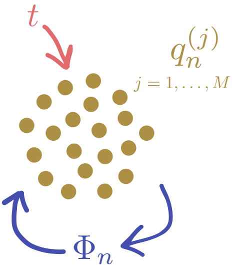
In this talk we will address the following question:

> *What are the criteria and mechanisms by which high-dimensional systems have linear response at macroscopic scales (for all practical purposes)?*



Method

Study a "simple complex system": globally coupled maps.



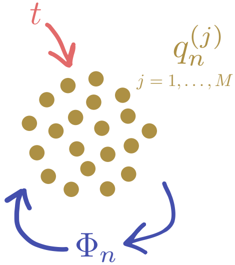
We derive LLN and CLT reductions for these maps as ensemble size $M \to \infty$, and study their linear response, considering different kinds of maps.
(These systems have rich dynamical and linear response behaviour!)

Model

$$q_n^{(j)} = f_t(q_{n-1}^{(j)}; \Phi_{n-1}),\ j = 1, \ldots, M$$

$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)})$$

Interested in dynamics/response of mean field Φ_n.



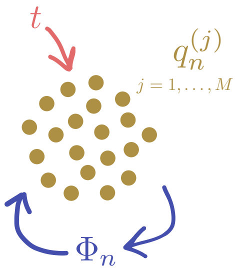
1. Thermodynamic limit: reduction

Formulate in terms of empirical measure of $q^{(j)}$s

$$\mu_n = \frac{1}{M} \sum_{j=1}^{M} \delta_{q_n^{(j)}}$$

so that system becomes

$$\mu_n = \mathcal{L}_{t,\Phi_n} \mu_{n-1}$$
$$\Phi_n = \int \phi \, \mathrm{d}\mu_n$$



where transfer operator $\mathcal{L}_{t,\Phi}$ pushes forward measures by $f_t(\,\cdot\,;\Phi)$.
In thermodynamic limit expect μ_n to be a measure typical of cocycle f_t (absolutely continuous if f uniformly expanding, etc.)

1. Thermodynamic limit: reduction

Under appropriate mixing conditions expect μ_n to be the time-varying physical invariant measure of cocycle

$$\mu_n^\infty = \lim_{k\to\infty} \mathcal{L}_{\Phi_{n-1},t} \cdots \mathcal{L}_{\Phi_{n-k},t}\rho_*$$

where ρ_* is any absolutely continuous probability measure. This gives us delay system in Φ:

$$\Phi_n = \int \phi \, \mathrm{d}\mu_n^\infty =: F_t(\Phi_{n-1}, \Phi_{n-2}, \ldots)$$

1. Thermodynamic limit: reduction

What are dynamics of

$$\Phi_n = F_t(\Phi_{n-1}, \Phi_{n-2}, \ldots)?$$

If f is exponentially mixing, then the effect of $\Phi_{n-k}, k \gg 1$ is negligible, i.e.

$$\Phi_n \approx F_t(\Phi_{n-1}, \Phi_{n-2}, \ldots, \Phi_{n-k_*})$$

Approximates finite-dimensional dynamics!

1. Thermodynamic limit: reduction

$$\Phi_n = F_t(\Phi_{n-1}, \Phi_{n-2}, \ldots)$$

- Partial derivative $\frac{\partial F_t}{\partial \Phi_{n-k}}$ is response of $\phi(q_n)$ to kicking mean-field input at times $n-k$.
- If cocycle $f_t(\,\cdot\,; \Phi_n)$ has summable response function (i.e. a linear response) then derivative is in $\ell^1(\mathbb{N})$ and we can do bifurcation theory on F.

1. Thermodynamic limit: reduction

$$\Phi_n = F_t(\Phi_{n-1}, \Phi_{n-2}, \ldots)$$

Suppose f has summable decay of response function coefficients.

- If $\Phi_n \equiv \bar{\Phi}^t$ is an equilibrium, then

$$\bar{\Phi}^t = F_t(\bar{\Phi}^t, \bar{\Phi}^t, \ldots) := G^t(\bar{\Phi}^t)$$

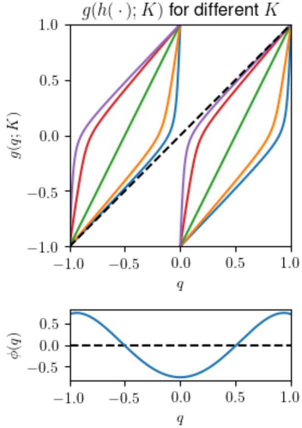
  - By implicit function theorem, $\frac{\mathrm{d}\bar{\Phi}^t}{\mathrm{d}t}$ exists when $\frac{\partial G_t(\Phi)}{\partial \Phi} \neq 1$
  - So coupled system has linear response (generically).
- Similar arguments hold for periodic, hyperbolic etc Φ-dynamics
- Non-hyperbolic dynamics (!) in mean-field Φ?

1. Thermodynamic limit: numerics

Choose

$$f_t(q;\Phi) = g(h(q), t\Phi)$$

for some g, h (see W. and Gottwald, '19). Have g is an analytic diffeo. and h is such that each $f_t(\cdot\,;\Phi)$ are analytic uniformly expanding.



1. Thermodynamic limit: numerics

Notice that we can write

$$\mu_n = \mathcal{L}_{t,\Phi_n}\mu_{n-1}$$
$$\Phi_n = \int \phi \, \mathrm{d}\mu_n$$

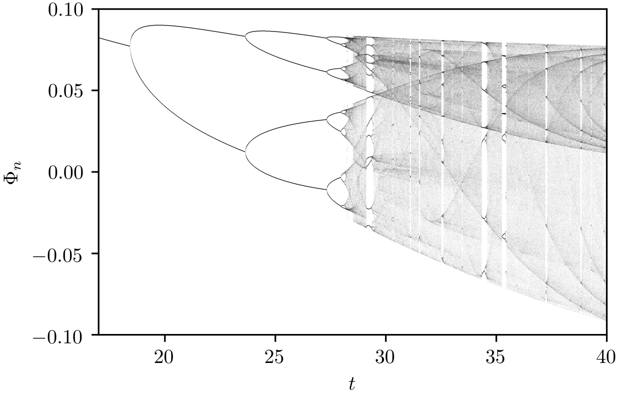
in terms of μ_n:

$$\mu_n = \mathcal{L}_{t,\int \phi \, \mathrm{d}\mu_{n-1}}\mu_{n-1} =: H_t(\mu_n)$$

We can approximate these dynamics *very* accurately using Chebyshev spectral methods,. e.g. in `Poltergeist.jl`.

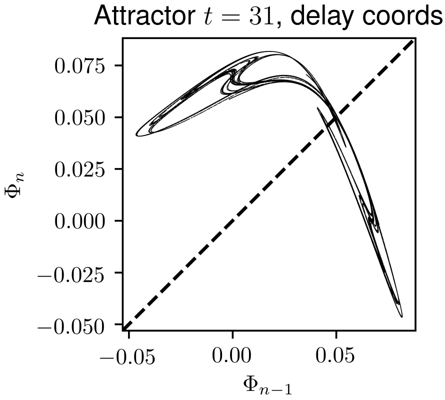
1. Thermodynamic limit: numerics

For large t we see period doubling bifurcation to chaos:



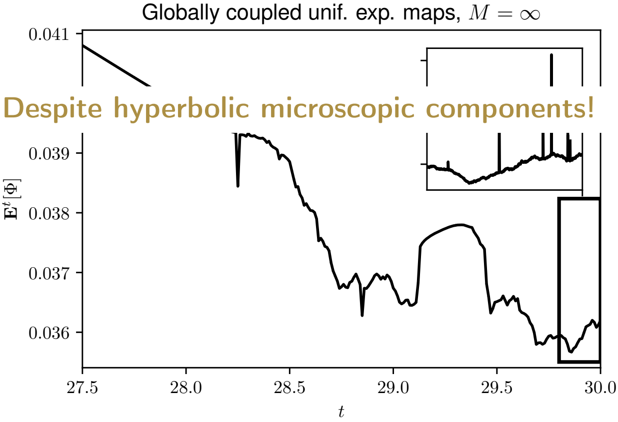
1. Thermodynamic limit: numerics

Chaotic Φ dynamics look unimodal (+ some contracting directions):



1. Thermodynamic limit: failure of LRT

In fact, a failure of linear response!



Despite hyperbolic microscopic components!

Can we find the culprit?

1. Thermodynamic limit: non-hyperbolicity

Using Chebyshev spectral methods, we numerically look for homoclinic tangencies in macroscopic dynamics: i.e. an orbit $\{\mu_n\}_{n\in\mathbb{N}}$ such that

- The orbit is homoclinic:

$$\lim_{n\to-\infty}\mu_n = \lim_{n\to\infty}\mu_n = \mu_*$$

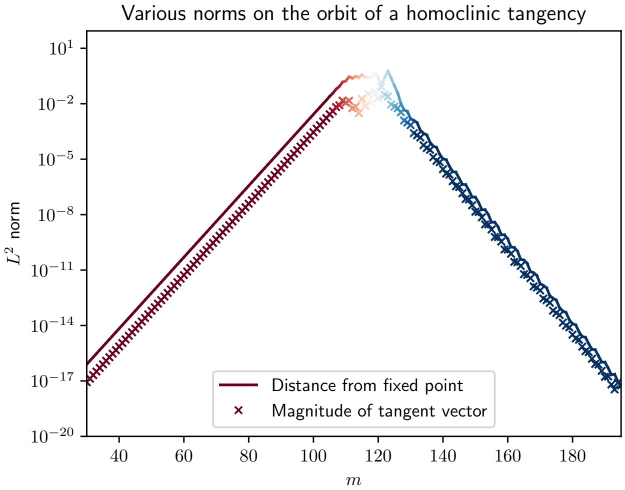
- There exist tangent vectors along this orbit $\delta\mu_n = J_{\mu_{n-1}}[G_t]\delta\mu_{n-1}$ such that

$$\lim_{n\to-\infty}\delta\mu_n = \lim_{n\to\infty}\delta\mu_n = 0,$$

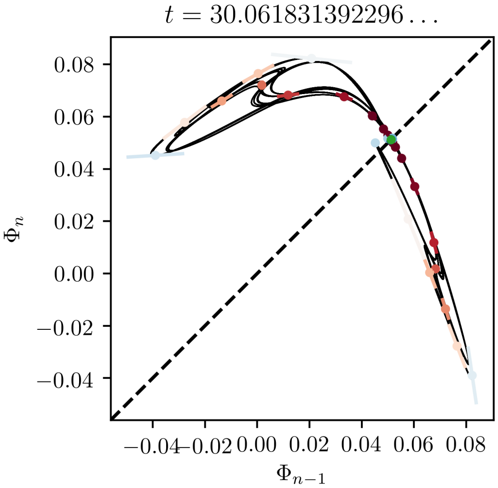
  i.e., $\delta\mu_n$ in both stable and unstable subspaces $\implies$ tangency of stable and unstable manifolds!

This would mean that the map is non-hyperbolic, with a nasty response

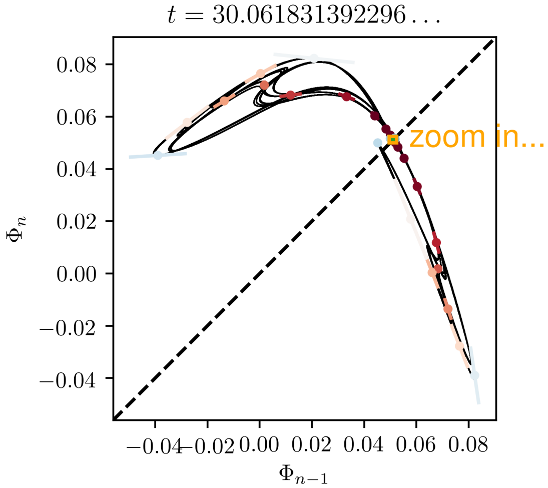
1. Thermodynamic limit: non-hyperbolicity



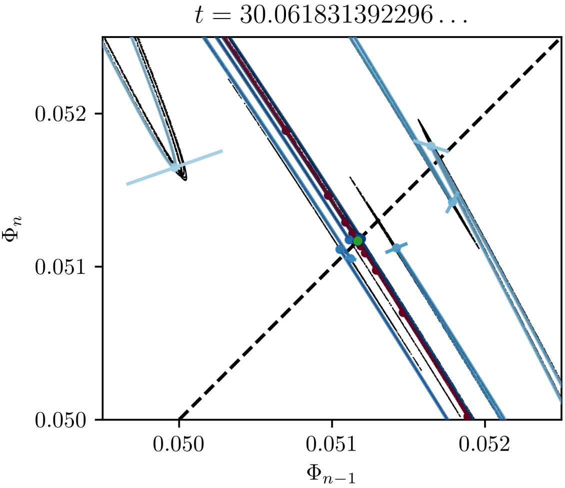
1. Thermodynamic limit: non-hyperbolicity



1. Thermodynamic limit: non-hyperbolicity



1. Thermodynamic limit: non-hyperbolicity



2. Finite size

In most applications expect an incomplete separation of scales: need to consider finite size effects.

Natural direction: central limit theorem.

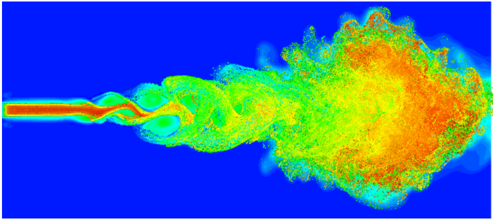
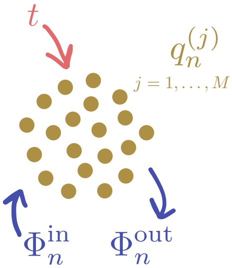


Figure: `http://mri-q.com`

2. Finite size

Remove the feedback loop:

$$\Phi_n^{\mathrm{in}} \text{ a pre-determined time series}$$
$$q_n^{(j)} = f_t(q_{n-1}^{(j)}; \Phi_{n-1}^{\mathrm{in}}),\ j = 1, \ldots, M$$
$$\Phi_n^{\mathrm{out}} = \frac{1}{M}\sum_{j=1}^{M} \phi(q_n^{(j)})$$



Here, the $q^{(j)}$ all evolve independently: they are samples from the time-varying map $f_t(\,\cdot\,; \Phi_n^{\mathrm{in}})$.

2. Finite size

If we consider the distribution of the system in the thermodynamic limit:

$$\Phi_n^{\mathrm{in}} \text{ a pre-determined time series}$$

$$\mu_n^\infty = \mathcal{L}_{t,\Phi_{n-1}^{\mathrm{in}}} \mu_{n-1}^\infty$$

$$\Phi_n^{\mathrm{out},\infty} = \int \phi \, \mathrm{d}\mu_n^\infty = F_t(\Phi_{n-1}^{\mathrm{in}}, \Phi_{n-2}^{\mathrm{in}}, \ldots)$$

Then for large M,

$$\Phi_n^{\mathrm{out}} = F_t(\Phi_{n-1}^{\mathrm{in}}, \Phi_{n-2}^{\mathrm{in}}, \ldots) + \frac{1}{\sqrt{M}} \zeta_n,$$

where ζ_n is a centered Gaussian random variable.

2. Finite size

Because the CLT correction ζ_n is generated from a sample of $\phi(q_n^{(j)})$, we can determine its covariance for $n \geq m$:

$$\mathrm{Cov}[\zeta_n, \zeta_m] = \mathrm{Cov}(\phi(q_n), \phi(q_m)) = V_t^{n-m}(\Phi_{n-1}^{\mathrm{in}}, \Phi_{n-2}^{\mathrm{in}}, \dots)$$

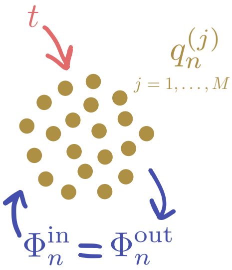
This has same decay of correlations as microscopic dynamics f.

2. Finite size

Ansatz: the self-coupled system can be modelled by setting $\Phi_n^{\mathrm{in}} = \Phi_n^{\mathrm{out}}$ after each step n.

This gives

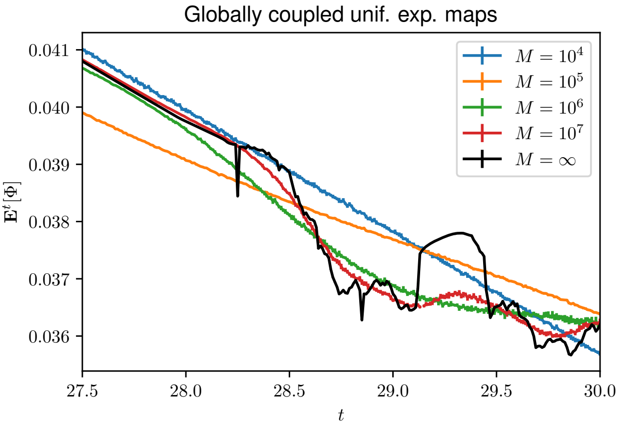
$$\Phi_n = F_t(\Phi_{n-1}, \Phi_{n-2}, \ldots) + \frac{1}{\sqrt{M}}\zeta_n,$$



where $\mathrm{Cov}[\zeta_n, \zeta_m] = V_t^{n-m}(\Phi_{n-1}, \Phi_{n-2}, \ldots)$.
This is a stochastic system.
$\implies$ we expect LRT

2. Finite size: numerical example 1

Consider the situation from the thermodynamic limit (i.e. uniformly expanding f)



Convergence of averages to thermodynamic limit.

2. Finite size: numerical example 2

What if the microscopic systems have a rough response when uncoupled (e.g. logistic maps)?
Can define dynamics in thermodynamic limit

$$\Phi_n = F_t(\Phi_{n-1}, \Phi_{n-2}, \ldots).$$

Effect of distant past $\Phi_{n-k}, k \gg 1$ is small, but derivatives cause problems:

a. Partial derivatives $\frac{\partial F_t}{\partial \Phi_{n-m}}$ are non-summable;
b. Partial derivative $\frac{\partial F_t}{\partial t}$ is undefined.

Don't expect smooth bifurcation theory or linear response.

2. Finite size: numerical example 2

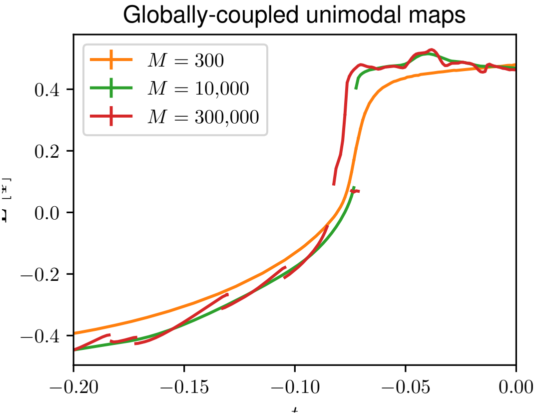
Fact: when forced by noise, logistic maps have a rough quenched response but a smooth annealed response.
This means that with noise, in some "annealed" sense,

a. Partial derivatives $\frac{\partial F_t}{\partial \Phi_{n-m}}$ become summable;
b. Partial derivative $\frac{\partial F_t}{\partial t}$ (linear response of cocycle) becomes defined.

Consequently expect LRT in our stochastic reduction.

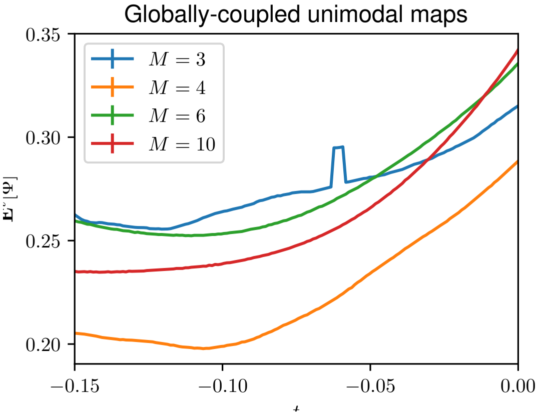
2. Finite size: numerical example 2

Breakdown of LRT occurs through bifurcations:



2. Finite size: numerical example 2

Linear response induced even for very small systems (e.g. $M = 4$):



May help explain success of LRT in locally-coupled systems.

3. Inhomogeneous microscopic subsystems

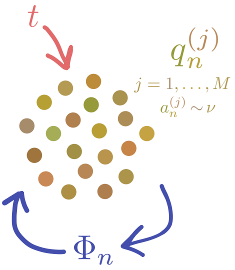
In the real world, there will be some variation between components:

3. Inhomogeneous microscopic subsystems

What if we have inhomogeneous subsystems?

$$q_n^{(j)} = f_t(q_{n-1}^{(j)}; \Phi_{n-1}, a^{(j)}),\ j = 1, \dots, M$$
$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)}),$$

where the $a^{(j)} \sim \nu$ are randomly selected parameters of the system.



3. Inhomogeneous microscopic subsystems

Two maps f and g share a topological conjugacy class if there exists a homeomorphism h such that

$$f = h \circ g \circ h^{-1}.$$

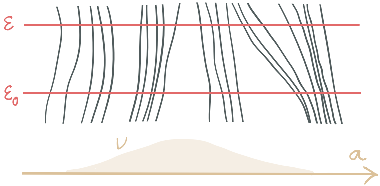
- Conjecture (Avila *et al* '03): unimodal maps' topological conjugacy classes have a uniformly analytic lamination of codimension 1.
- Typically, linear response fails only if perturbations change the system's topological conjugacy class (Baladi '14).

3. Inhomogeneous microscopic subsystems

So: if ensemble contains all top. conj. classes (via variation in $a^{(j)}$), then we can find a conjugacy between microscopic dynamics at t_0 and at t by changing a:

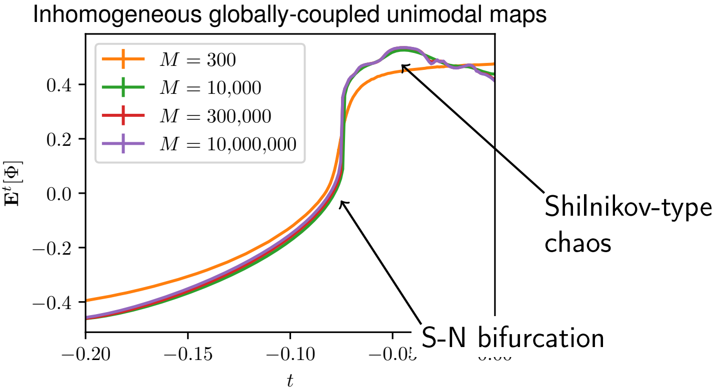
$$f_t(\,\cdot\,;\Phi, a_0) = h \circ f_{t_0}(\,\cdot\,;\Phi, a_t(a_0, t_0)) \circ h^{-1},$$

where the parameter $a_t(a_0, t_0)$ lies in the support of ν, and a_t is differentiable in t.
If ν has a C^1 density, this gives us LRT.



3. Inhomogeneous microscopic subsystems

Upshot: same as if individual subsystems have LRT (i.e. well-posed thermodynamic limit, etc.)



Conclusions

Studied globally coupled systems via models for macroscopic dynamics for large/infinite M

- At finite size have emergent stochastic effects, which induces linear response
- In thermodynamic limit linear response is more complicated:
  - Depends on microscopic systems' LRT properties
  - Also requires "nice" macroscale dynamics
  - Surprisingly: can be non-hyperbolic and violate LRT
- Parameter variation between subsystems helps generate LR at macroscale.

Further details

Wormell, C.L. and Gottwald, G.A., 2019. Linear response for macroscopic observables in high-dimensional systems. `arXiv:1907.13490`.

Wormell, C.L., forthcoming. Homoclinic tangencies in the macroscopic dynamics of a globally coupled system.

1. Thermodynamic limit: numericals

Specifically for $q \in [-1, 1]$:

$$h(q) = (2q \mod 2) - 1$$

$$g(q, t\Phi) = \frac{q + K\left(1 - \sqrt{0.03(1 - 0.97K^2) + 0.97(q + K_n)^2}\right)}{1 - 0.97K^2}$$

where $K = \tanh(t\Phi - 2)$. Also, $\phi(q) = -\frac{23}{30} + \frac{7}{2}q^2 - 2q^4$.