# The homology of symmetric groups and the algebra of covering spaces

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In this work we investigate the homology of the symmetric groups from an algebraic and geometric viewpoint. Homology is taken modulo a prime number $p$ and we shall discuss the geometry in the case $p = 2$ only. We are guided by an analogy between homology groups and character groups and we shall compare the direct sum HS of the homology of the symmetric groups with the direct sum RS of their character groups.

Recall that RS is naturally isomorphic via the Frobenius correspondence to the ring of symmetric functions. The character group of an individual symmetric group is then sitting as a homogeneous component of the ring of symmetric functions. This illustrates the fact that the whole can be simpler than its parts! There is another neat description of RS using the concept of lambda-ring due to Grothendieck. Every character ring has lambda operations mirroring exterior powers of linear representations and the elements of RS can be viewed as operations on character rings. The ring RS turns out to be the free lambda-ring on one generator. The substitution of operations is the classical plethism of symmetric functions. The purpose of our talk is to produce a similar picture for the direct sum HS of the homology of the symmetric groups. We introduce the concept of $Q$-ring and prove that HS is the free $Q$-ring on one (even) generator. Here the role of symmetric functions is played by modular invariants (Dickson and Mui invariants). We explain the relation with Steenrod and Dyer-Lashof operations. When $p = 2$ homology classes can be modelled with bordism classes and this suggests considering the direct sum NS of the bordism groups of the symmetric groups. Its elements are cobordism classes of finite covering spaces of manifolds. We introduce the concept of $Q$-ring relative to a formal group law and show that NS is the free such structure on one generator. The operation of substitution (plethism) of finite covering spaces has a direct geometric description.