Before moving on to more theory, let us calculate some examples. Unfortunately, verifying elegant theoretical results in practice often involves long calculations.

Let us start with a degree 1 representation of the group of symmetries of a square. Labelling the vertices of the square 1, 2, 3 and 4 (cyclically), and identifying its symmetries with pemutations of the vertices, we can easily write down the eight symmetries. The rotation symmetries are the powers of the 4-cycle $(1\,2\,3\,4)$, and multiplying these four elements by $(1\,3)$ gives the four reflection symmetries. Since we know that the product of two reflections is a rotation, and the product of two rotations is a rotation, while the product of a reflection and a rotation is a reflection, it follows that there is a representation of the group which maps the reflections to -1 and the rotations to $+1$.

So we have eight permutations which form a subgroup D of $G = S_4$, the group of all permutations of $\{1, 2, 3, 4\}$, and we have a representation $\rho\colon D \to \mathrm{GL}(1, \mathbb{C})$. We should be able to form an induced representation. For this we first need a system of coset representatives. Now $[G : D] = 24/8 = 3$; so we need three coset representatives. An easy way to find three suitable permutations is to find a subgroup X of G with three elements. The subgroup $H \cap X$ must then be trivial (since its order must divide both 3 and 8), and so if x, y are distinct elements of X then the cosets xH and yH are distinct (since $x^{-1}y \notin H$). So we can take $x_1 = \mathrm{id}$, $x_2 = (1\,2\,3)$ and $x_3 = (1\,3\,2)$ as the coset representatives.

Now let us calculate the matrices which will represent some randomly chosen elements of S_4 in the induced representation ρ^G. Consider $g = (1\,4\,3)$, for example. The cosets gx_1H, gx_2H, gx_3H must equal x_1H, x_2H, x_3H in some order. Now we have to calculate. First, $x_2^{-1}gx_1 = (1\,3\,2)(1\,4\,3) = (1\,4\,2) \notin D$. Also $x_1^{-1}gx_1 = g \notin D$. So it must be that $x_3^{-1}gx_1 \in H$, and indeed calculation yields $x_3^{-1}gx_1 = (1\,4)(2\,3)$, which is the reflection in the perpendicular bisector of the sides 1-4 and 2-3 of the square. So $\rho(x_3^{-1}gx_1) = -1$, and this is the $(3, 1)$ entry of the matrix $\rho^G(g)$. The $(2, 1)$ and $(1, 1)$ entries are 0. Moving on to the second column, we know that the $(3, 2)$ entry must be zero, since the $(3, 1)$ entry is nonzero, and since $x_2^{-1}gx_2$ is a 3-cycle (and hence not in H) we conclude that the $(1, 2)$ entry must be nonzero. We readily find that $x_1^{-1}gx_2 = (1\,4\,3)(1\,2\,3) = (1\,2)(3\,4)$, another reflection. And finally the nonzero entry in the 3rd column must be in the 2nd row (since the other two rows are taken), and the entry must be $\rho((1\,3\,2)(1\,4\,3)(1\,3\,2)) = \rho((1\,3)(2\,4)) = 1$ (since $(1\,3)(2\,4)$ is a rotation). So we obtain the matrix

$$\rho^G((1\,4\,3)) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let us next calculate $\rho^G((1\,2))$. We find that $x_2^{-1}(1\,2)x_2 = (1\,3)$ (one of the reflections in D), $x_1^{-1}(1\,2)x_3 = (1\,3)$ and $x_3^{-1}(1\,2)x_1 = (1\,3)$. So

$$\rho^G((1\,2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The logical thing to do next is to check that $\rho^G((1\,2)\rho^G((1\,4\,3)) = \rho^G((1\,2)(1\,4\,3)$. By what we have done so far the left hand side is

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

So as $(1\,2)(1\,4\,3) = (1\,4\,3\,2)$ we only have to check that

$$\rho^G((1\,4\,3\,2)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now $\rho(x_1^{-1}(1\,4\,3\,2)x_1) = \rho((1\,4\,3\,2)) = 1$, which confirms that the $(1,1)$ entry is correct. Similarly $\rho(x_3^{-1}(1\,4\,3\,2)x_2) = \rho(1\,2\,3\,4) = 1$ and $\rho(x_2^{-1}(1\,4\,3\,2)x_3) = \rho((2\,4)(1\,3)) = -1$, which also check.

If we had started with a representation of D of degree 2, or more, the calculations would have been very similar; the main difference would just be the size of the resulting matrices. For example, there is a representation $\sigma\colon D \to \mathrm{GL}(2,\mathbb{C})$ defined by

$$\sigma((1\,2\,3\,4)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma((1\,2)(3\,4)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and if we use the same coset representatives as above to compute σ^G we find that

$$\sigma^G((1\,4\,3)) = \begin{pmatrix} 0 & \sigma((1\,2)(3\,4)) & 0 \\ 0 & 0 & \sigma((1\,3)(2\,4)) \\ \sigma((1\,4)(2\,3)) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

as $\sigma((1\,3)(2\,4)) = \sigma((1\,2\,3\,4))^2 = -I$ and $\sigma((1\,4)(2\,3)) = \sigma((1\,3)(2\,4))\sigma((1\,2)(3\,4)) = -\sigma((1\,2)(3\,4))$.

Returning to our first example, let us now calculate the character of ρ^G. Recall that $G = S_4$ has five conjugacy classes $\mathcal{C}_i$ corresponding to the five possible cycle types for permutations of $\{1,2,3,4\}$. As representatives of these classes we can take the following elements g_i:

$$g_1 = \mathrm{id}, \quad g_2 = (1\,2), \quad g_3 = (1\,2\,3), \quad g_4 = (1\,2\,3\,4) \quad \text{and} \quad g_5 = (1\,2)(3\,4).$$

If h_i is the number of elements in the class $\mathcal{C}_i$, it is easy shown that

$$h_1 = 1, \quad h_2 = 6, \quad h_3 = 8, \quad h_4 = 6 \quad \text{and} \quad h_5 = 3.$$

In order to calculate the induced character using the formula from the end of Lecture 14 we need to also know the conjugacy classes $\mathcal{D}_j$ of $\mathcal{D}$ and all the containment relations $\mathcal{D}_j \subseteq \mathcal{C}_i$. Now in fact there are five $\mathcal{D}_j$'s. Two of these have one element each: $\mathcal{D}_1 = \{\mathrm{id}\}$ and $\mathcal{D}_2 = \{(1\,3)(2\,4)\}$. (Observe that $(1\,3)(2\,4)$ is the half-turn, corresponding to the linear transformation which is -1 times the identity.) The other three classes have two elements each: firstly, $\mathcal{D}_3 = \{(1\,2\,3\,4), (1\,4\,3\,2)\}$ (the remaining two rotations); next, $\mathcal{D}_4 = \{(1\,3), (2\,4)\}$ (the reflections in the two diagonals of the square); finally $\mathcal{D}_5 = \{(1\,2)(3\,4), (1\,4)(2\,3)\}$ (the reflections in perpendicular bisectors of the sides). The character χ of ρ takes the value $+1$ on elements of classes $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, and -1 on elements of $\mathcal{D}_4$ and $\mathcal{D}_5$. (Of course in this case the character χ is just the same as the representation ρ, since the degree of the representation is 1.)

By the formula from the end of Lecture 14, the value the induced character χ^G takes at an element $g \in G$ is given by

$$\chi^G(g) = \frac{|G|}{|H|}\sum_j \frac{q_j}{h}\chi(l_j) = 3\sum_j \frac{q_j}{h}\chi(l_j) \tag{1}$$

where h is the size of the conjugacy class of G containing g, the l_j are representatives of the conjugacy classes of D that are contained in the G-conjugacy class of g and the q_j are the sizes of these classes. Now the class $\mathcal{C}_3$ of G does not contain any elements of D; so the sum in Eq. (1) is empty if $g \in \mathcal{C}_3$, and thus $\chi^G(g) = 0$ in this case. Classes $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_4$ of G each contain only one conjugacy class of D. So

$$\begin{aligned} \chi^G(g_1) &= 3\frac{1}{1}\chi(1) = 3 \\ \chi^G(g_2) &= 3\frac{2}{6}\chi((1\,2)) = -1 \\ \chi^G(g_4) &= 3\frac{2}{6}\chi((1\,2\,3\,4)) = 1. \end{aligned}$$

It remains to calculate $\chi^G(g_5)$. In this case the conjugacy class of G (namely $\mathcal{C}_5$) contains two conjugacy classes of D (namely $\mathcal{D}_2$ and $\mathcal{D}_5$). So the formula yields

$$\chi^G(g_5) = 3(\frac{1}{3}\chi((1\,3)(2\,4)) + \frac{2}{3}\chi((1\,2)(3\,4))) = 3(\frac{1}{3} - \frac{2}{3}) = -1$$

Although the character of D that we started with is obviously irreducible, having degree 1, the induced character χ^G does not have to be. We can determine how close it is to being irreducible by calculating its inner product with itself. We find

$$\begin{aligned} (\chi^G, \chi^G) &= \frac{1}{24}\sum_{g\in G} |\chi^G(g)|^2 \\ &= \frac{1}{24}\sum_{i=1}^{5} h_i|\chi^G(g_i)|^2 \\ &= \frac{1}{24}(1 \times 9 + 6 \times 1 + 0 + 6 \times 1 + 3 \times 1) \\ &= 1. \end{aligned}$$

Thus χ^G is in fact irreducible after all.

Let us also calculate the character of the induced representation σ^G mentioned above. Writing ψ for the character of σ, the following table gives the character values on elements from the various classes of D.

classes	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$
$\psi(x)$	2	-2	0	0	0

As for χ^G, it is immediate that $\psi^G(g_3) = 0$. Also as for χ^G the value that ψ^G takes on g_1, g_2 and g_4 is found by multiplying $\psi(g_1)$, $\psi(g_2)$ and $\psi(g_4)$ by the appropriate ratios, which are respectively 3, 1 and 1. And finally

$$\psi^G(g_5) = 3(\frac{1}{3}\psi((1\,3)(2\,4) + \frac{2}{3}\psi((1\,2)(3\,4))) = -2.$$

So the values of the induced character are as follows

classes	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
$\psi^G(g)$	6	0	0	0	-2

We readily find that $(\psi^G, \psi^G) = \frac{1}{24}(36 + 3 \times 4) = 2$, and from this it follows that ψ^G must be a sum of two irreducible characters of G. We can also check that $(\psi^G, \chi^G) = 1$, so that χ^G is in fact one of the irreducible constituents of ψ^G. The difference $\psi^G - \chi^G$ must therefore be another irreducible character of G.