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THE UNIVERSITY OF SYDNEY
FACULTY OF SCIENCE

MATH3010

Information Theory – SAMPLE EXAM ONLY

Lecturer: R. B. Howlett **Time allowed: two hours**

Questions carry equal credit. No books or notes may be used in this examination. Calculators may be used. In each part of Questions 3 & 4 only one of the alternatives is correct. Decide which one this is and place a mark in the corresponding box. In multiple choice questions incorrect or multiple selections score zero.

This examination paper has 5 pages and 8 questions

1. (*i*) A random variable X takes the values 0 and 1 with equal probabilities. The *variance* $\sigma^2(X)$ is

☐ $\frac{1}{4}$. ☐ $\frac{1}{2}$. ☐ 1. ☐ $\sqrt{2}$.

(*ii*) Which of the following statements about the expectations of random variable X and Y is *always* true?

(a) $E(XY) = E(X)E(Y)$.

(b) $E(X^2) \geq (E(X))^2$.

(c) $E(X) > 0$ implies that $X \geq 0$ with probability 1.

(d) $E(X) = 0$ implies that $X = 0$ with probability 1.

(*iii*) Which of the following statements is *always* true?

(a) If $H(X \mid Y) = H(X) - H(Y)$ then X and Y are independent.

(b) If $H(X \mid Y) = 0$ then X and Y are independent.

(c) If the mutual information $I(X;Y)$ is zero then X and Y are independent.

(d) If $H(X,Y) = 0$ then X and Y are independent.

(*iv*) Suppose that an experiment has the two outcomes A and B. Let p be the distribution with $p(A) = \frac{1}{3}$ and $p(B) = \frac{2}{3}$. Let q be the distribution with $q(A) = \frac{2}{3}$ and $q(B) = \frac{1}{3}$. The *cross entropy* $D(p \,||\, q)$ (in bits) is:

☐ $-\frac{1}{3}$, ☐ 0, ☐ $\frac{1}{3}$, ☐ $\frac{2}{3}$.

2. (*i*) Which of the following sequences of integers could be the codeword lengths for a binary prefix code?

(a) 1, 2, 2, 3,

(b) 1, 2, 3, 3,

(c) 1, 2, 2, 2,

(d) 2, 2, 2, 2, 2.

(*ii*) Which of the following sets of codewords could be the Huffman code for some 4 symbol source alphabet?

(a) 01, 10, 00, 111,

(b) 0, 10, 110, 111,

(c) 1, 01, 10, 001,

(d) 0, 110, 111, 101.

(*iii*) A Markov process has two states A and B. When in state A it *always* changes to state B, and *vice-versa*. Then

(a) The information rate of the process is 1 bit,

(b) There is no initial probability distribution which makes the process stationary,

(c) The process is stationary for an initial distribution which assigns equal probabilities to both A and B,

(d) The process is stationary for any initial probability distribution,

(*iv*) Which of the following functions is *not* convex on the specified range?

(a) $|x|$ for $x \in \mathbb{R}$,

(b) $\ln x$ for $x > 0$,

(c) $x^2 - x - 1$ for $x \in \mathbb{R}$,

(d) e^x for $x \in \mathbb{R}$,

3. (*i*) The following table shows joint and marginal distributions for two random variables X and Y, with some entries missing.

	x_1	x_2	$p(y)$
y_1	0.5	?	0.6
y_2	?	?	?
$p(x)$	?	0.2	

Make a copy of the table, and fill in the missing entries.

(*ii*) Define the *expectation* $E(X)$ and *variance* $\sigma^2(X)$ of a random variable X (assumed to take only finitely many values).

(*iii*) Suppose that X and Y are *independent* random variables. From your definition show that

$$\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y)\,.$$

To simplify the proof, you may assume that both X and Y have zero expectation.

4. (*i*) What is the definition of the *entropy* $H(X)$ of a probability space (X, p)?

(*ii*) Under what conditions is the entropy $H(X)$ equal to zero?

(*iii*) Let $p(x, y)$ be the joint probability distribution of the two random variables X and Y. Define the *conditional entropy* $H(X \mid Y)$ in terms of the joint distribution and associated conditional probabilities.

(*iv*) Prove the formula:

$$H(X, Y) = H(X \mid Y) + H(Y)\,.$$

5. (*i*) An alphabet contains the five symbols $\{A,\, B,\, C,\, D,\, E\}$, which appear with probabilities

$$p(A) = \tfrac{1}{2}, \quad p(B) = \tfrac{1}{8}, \quad p(C) = \tfrac{1}{8}, \quad p(D) = \tfrac{1}{8}, \quad p(E) = \tfrac{1}{8}\,.$$

Design a binary Huffman code for this alphabet.

(*ii*) Which of the following functions are convex on the specified domains? Briefly justify your answer in each case by reference to any results proved in the course.

(a) $\sin(x)$ for $0 \le x \le \pi$,

(b) $|x|$ for $x \in \mathbb{R}$,

(c) e^x for $x \in \mathbb{R}$.

6. A gambler has the opportunity to bet repeatedly on the occurrence of a certain event, which is known to occur with probability 0.6, and returns \$2 for each \$1 bet (for a profit of \$1). Suppose that the gambler bets a fraction b of total funds each time, where $0 \leq b \leq 1$.

(*i*) Write down a formula for the expected rate of growth of the gambler's funds.

(*ii*) Find the value of b which maximizes this expected growth rate.

(*iii*) Calculate the expected growth rate in this case.

Give your answers correct to three decimal places.

7. (*i*) Let f be a real-valued function defined on an interval $[a, b]$ of the real line. Suppose that f is strictly convex, in the sense that for each point x of the interval there is a linear function h with $f(x) = h(x)$ and $h < f$ elsewhere in the interval.

(a) Show that

$$f((1-t)x + ty) < (1-t)f(x) + tf(y)$$

for any two distinct points x and y in the interval, and $0 < t < 1$.

(b) Hence, or otherwise, prove that if x is a point in the interval where f takes its minimum value, then x is the *unique* point with this property.

(*ii*) Give an example of a strictly convex function f defined on the real line $\mathbb{R}$ for which there is *no* point x for which $f(x) < f(y)$ for all points $x \neq y$ in $\mathbb{R}$. Briefly justify your example.

8. (*i*) Let $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ be two probability distributions on the same sample space, which we may take to be the set of integers i with $1 \leq i \leq n$. Define the *cross-entropy* $D_e(p \,||\, q)$ of the two distributions (to base e).

(*ii*) Suppose that a successful bet on the occurrence of event i returns T_i units of wealth (including the original stake) for each unit wagered. Otherwise the stake is lost. Let p_i be the probability of event i.

Suppose that on each trial a gambler allocates all available funds, betting a fraction b_i of total funds on event i. Write down a formula for the expected rate of increase of the gambler's wealth.

How should the b_i be chosen in order to maximise this expected rate of increase? Prove that your strategy gives the maximum value. Any properties of the cross entropy proved in the course may be used without proof, if carefully stated.