Tutorial 11

1. Let $\theta: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\theta\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Calculate the matrix of θ relative to the bases

$$\boldsymbol{d} = \left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}\right) \quad \text{and} \quad \boldsymbol{c} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

of $\mathbb{R}^3$ and $\mathbb{R}^2$.

2. (*i*) Let $\phi: \mathbb{R}^2 \to \mathbb{R}$ be defined by $\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Calculate the matrix of ϕ relative to the bases

$$\boldsymbol{c} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad \boldsymbol{b} = (-1)$$

of $\mathbb{R}^2$ and $\mathbb{R}$.

(*ii*) With ϕ as in (*i*) and θ as in Exercise 1 calculate $\phi\theta$ and its matrix relative the two given bases. Hence verify that $\mathrm{M}_{\boldsymbol{bd}}(\phi\theta) = \mathrm{M}_{\boldsymbol{bc}}(\phi)\,\mathrm{M}_{\boldsymbol{cd}}(\theta)$.

3. Suppose that $\theta: \mathbb{R}^6 \to \mathbb{R}^4$ is a linear transformation with kernel of dimension 2. Is θ surjective?

4. For each of the following linear transformations calculate the dimensions of the kernel and image, and check that your answers are in agreement with the Main Theorem on Linear Transformations.

(*i*) $\theta: \mathbb{R}^4 \to \mathbb{R}^2$ given by $\theta\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 & 5 \\ 1 & -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$.

(*ii*) $\theta: \mathbb{R}^2 \to \mathbb{R}^3$ given by $\theta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

(*iii*) $\theta: V \to V$ given by $\theta(p(x)) = p'(x)$, where V is the space of all polynomials over $\mathbb{R}$ of degree less than or equal to 3.

5. Is it possible to find a 3×2 matrix A, a 2×2 matrix B and a 2×3 matrix C such that

$$ABC = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 0 \\ 4 & 0 & 0 \end{pmatrix}?$$

6. Let V and W be finitely generated vector spaces of the same dimension and let $\theta: V \to W$ be a linear transformation. Use the Main Theorem on Linear Transformations to prove that θ is injective if and only if it is surjective.