THE UNIVERSITY OF SYDNEY

MATH2008 Introduction to Modern Algebra

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2008/)

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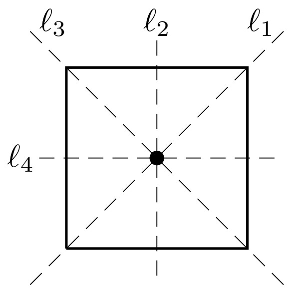
## Tutorial 6

The square shown below has four reflection symmetries, corresponding to the four "axes of symmetry" $\ell_1$, $\ell_2$, $\ell_3$ and $\ell_4$. It also has four rotation symmetries. Write $\sigma$ for the reflection in $\ell_1$ and $\rho$ for the anticlockwise rotation through $90^0$ about the centre. Let $G$ be the group of all symmetries of the square.




**1.** Describe the element $\rho^3\sigma$ geometrically. Find expressions in terms of $\rho$ and $\sigma$ for all 8 elements of $G$, and describe them all geometrically (as reflections and rotations).

*Solution.*

Number the locations of the vertices 1, 2, 3, 4, in anticlockwise order starting from the top right-hand corner. Then $\rho$ corresponds to the 4-cycle $(1,2,3,4)$ and $\sigma$ to the transposition $(2,4)$. To find out what $\rho^3\sigma$ is we can either multiply these permutations, or else follow what happens to each vertex when one first performs the rotation $\rho^3$ and then the reflection $\sigma$. In fact, $\rho^3$ moves the contents of location 1 to location 4, and 4 to 3, 3 to 2 and 2 to 1. Follow this by $\sigma$, which swaps the contents of locations 2 and 4, and the net effect is to take the contents of location 1 to location 2, and 2 to 1, 3 to 4 and 4 to 3. Alternatively, computing the permutation product $(1,2,3,4)(2,4)$ gives the answer $(1,2)(3,4)$. (Indeed, this is essentially the same calculation.) We conclude that $\rho^3\sigma$ is the reflection in $\ell_2$.

It turns out that computing all the products $\rho^i$ and $\rho^i\sigma$ for $i = 1, 2, 3, 4$ gives all eight elements of $G$. The full list of elements of $G$ is as follows:

| | |
|---|---|
| $e$ : | the identity (do nothing) |
| $\rho$ : | the anticlockwise rotation through 90° |
| $\rho^2$ : | the rotation through 180° |
| $\rho^3$ : | the clockwise rotation through 90° |
| $\sigma$ : | the reflection in $\ell_1$ |
| $\rho\sigma$ : | the reflection in $\ell_4$ |
| $\rho^2\sigma$ : | the reflection in $\ell_3$ |
| $\rho^3\sigma$ : | the reflection in $\ell_2$ |

**2.** Construct the multiplication table for the set of symmetries

$$H = \{e, \rho, \rho^2, \rho^3\}$$

and verify that $H$ is an abelian subgroup of $G$. (Here $e$ denotes the identity element.)

*Solution.*

| | $e$ | $\rho$ | $\rho^2$ | $\rho^3$ |
|---|---|---|---|---|
| $e$ | $e$ | $\rho$ | $\rho^2$ | $\rho^3$ |
| $\rho$ | $\rho$ | $\rho^2$ | $\rho^3$ | $e$ |
| $\rho^2$ | $\rho^2$ | $\rho^3$ | $e$ | $\rho$ |
| $\rho^3$ | $\rho^3$ | $e$ | $\rho$ | $\rho^2$ |

The product $\rho^i\rho^j = \rho^{i+j}$, and in those cases in which $i + j \geq 4$ we use the fact that $\rho^4 = e$ to conclude that $\rho^i\rho^j = \rho^{i+j-4}$. This gives us the above table, and also shows that $H$ is closed under multiplication. So **SG1** holds. The identity is one of the elements of $H$; so **SG2** holds. And the inverse of every element of $H$ is in $H$: the inverse of $e$ is $e$, the inverse of $\rho^2$ is $\rho^2$, and $\rho$ and $\rho^3$ are inverses of each other. So **SG3** also holds, and so $H$ is a subgroup of $G$.

The net effect of a rotation through $\alpha$ followed by a rotation through $\beta$ (about the same point) is a rotation through $\alpha + \beta$, and doing the rotation through $\beta$ first and then the rotation through $\alpha$ gives the same result. In the present example this says that $\rho^i\rho^j = \rho^j\rho^i$ for all $i$ and $j$. This is obvious anyway, since both equal $\rho^{i+j}$.

A finite group is abelian if and only if its multiplication table is symmetric about the main diagonal, since the entry in the row labelled by $x$ and the column labelled by $y$ is $xy$, while the entry in the row labelled by $y$ and the column labelled by $x$ is $yx$ (and transposing the table interchanges these positions). The table above is indeed symmetric.

**3.** Repeat Question 2 with the following sets in place of $H$.

(*i*) $K = \{e, \rho^2, \sigma, \rho^2\sigma\}$,

(*ii*) $L = \{e, \rho^2, \rho\sigma, \rho^3\sigma\}$.

*Solution.*

It is easily checked that the half turn $\rho^2$ and the reflection $\sigma$ commute with each other: $\rho^2\sigma = \sigma\rho^2$. Combined with the facts that $\rho^4$ and $\sigma^2$ are both equal to the identity $e$, this enables us to write down the multiplication tables for $K$ and $L$. Note that both are closed under multiplication and contain the identity. Furthermore, all the elements are their own inverses. So $K$ and $L$ are subgroups, and abelian, as the tables are symmetric.

| | $e$ | $\rho^2$ | $\sigma$ | $\rho^2\sigma$ |
|---|---|---|---|---|
| $e$ | $e$ | $\rho^2$ | $\sigma$ | $\rho^2\sigma$ |
| $\rho^2$ | $\rho^2$ | $e$ | $\rho^2\sigma$ | $\sigma$ |
| $\sigma$ | $\sigma$ | $\rho^2\sigma$ | $e$ | $\rho^2$ |
| $\rho^2\sigma$ | $\rho^2\sigma$ | $\sigma$ | $\rho^2$ | $e$ |

| | $e$ | $\rho^2$ | $\rho\sigma$ | $\rho^3\sigma$ |
|---|---|---|---|---|
| $e$ | $e$ | $\rho^2$ | $\rho\sigma$ | $\rho^3\sigma$ |
| $\rho^2$ | $\rho^2$ | $e$ | $\rho^3\sigma$ | $\rho\sigma$ |
| $\rho\sigma$ | $\rho\sigma$ | $\rho^3\sigma$ | $e$ | $\rho^2$ |
| $\rho^3\sigma$ | $\rho^3\sigma$ | $\rho\sigma$ | $\rho^2$ | $e$ |

**4.** Show that the group of complex numbers of modulus 1 is a subgroup of the group of all non-zero complex numbers (with multiplication as the group operation).

*Solution.*

Let $\mathbb{T} = \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$. For all complex numbers $z$ and $w$ we have $|zw| = |z|\,|w|$. So if $|z| = |w| = 1$, then $|zw| = 1$. Thus $\mathbb{T}$ is closed under multiplication. Certainly $|1| = 1$ and so the the identity 1 belongs to $\mathbb{T}$. If $|z| = 1$, then $|1/z| = 1/|z| = 1$; therefore $\mathbb{T}$ contains the inverse of each of its elements. Hence $\mathbb{T}$ is a subgroup.

**5.** Prove that $H = \{1, -1, i, -i\}$ is a subgroup of the group in Question 4. Determine all subgroups of $H$.

*Solution.*

| | $1$ | $-1$ | $i$ | $-i$ |
|---|---|---|---|---|
| $1$ | $1$ | $-1$ | $i$ | $-i$ |
| $-1$ | $-1$ | $1$ | $-i$ | $i$ |
| $i$ | $i$ | $-i$ | $-1$ | $1$ |
| $-i$ | $-i$ | $i$ | $1$ | $-1$ |



Every element of $H$ has modulus 1; so $H$ is a subset of the group $\mathbb{T}$ of Question 4. Using $i^2 = -1$, it is straightforward to derive the above multiplication table, and thereby see that $H$ is closed under multiplication. It certainly contains the identity element of $\mathbb{T}$ (namely 1), and contains the inverses of all of its elements: $1^{-1} = 1$, $(-1)^{-1} = -1$, $i^{-1} = -i$, $(-i)^{-1} = i$.

Every group is a subgroup of itself, and every group has the trivial subgroup whose only element is the identity. So $\{1\}$ and $H$ are subgroups of $H$. It is easily checked that $\{1, -1\}$ is also a subgroup—the conditions SG1, SG2 and SG3 are obviously satisfied. To see that there are no more, suppose that $K$ is a subgroup. By definition we must have $1 \in K$. If $i \in K$ then closure forces $-1 = i^2$ and $-i = i^3$ to be elements of $K$. So $K$ contains every element of $H$; that is, $K = H$. Similarly, if $-i \in K$ then $K$ also contains $-1 = (-i)^2$ and $i = (-i)^3$, and again $K = H$. If neither $i$ nor $-i$ is in $K$ then $K = \{1\}$ or $\{1, -1\}$, depending on whether or not $-1 \in K$. So the only subgroups of $H$ are the ones we have listed.

**6.** Prove that the set

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

is an abelian subgroup of the group of all invertible $2 \times 2$ real matrices.

*Solution.*

Everything that needs to be verified follows from the equation

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t+u & 1 \end{pmatrix}.$$

Firstly, this immediately shows that $U$ is closed under multiplication. Taking $t = 0$ shows that the identity element is in $U$. Putting $u = -t$ in the above equation, we see that

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix},$$

and since this is also in $U$ we conclude that $U$ contains the inverse of all of its elements. Hence $U$ is a subgroup of the group of invertible $2 \times 2$ matrices.

We see that $U$ is abelian, since for all $t$, $u \in \mathbb{R}$ we have

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t+u & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u+t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$