The University of Sydney

MATH3906 Representation Theory

(http://www.maths.usyd.edu.au/u/UG/IM/MATH3906/)

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Tutorial 5

1. Use orthogonality of coordinate functions to prove that if χ and ϕ are characters of irreducible complex representations of G then

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\phi(g)=\begin{cases}1 & \text{if } \chi=\phi\\ 0 & \text{if } \chi\neq\phi.\end{cases}$$

(Hint: Choose a full set of irreducible unitary representations $R^{(k)}$ of G, as in Lecture 9. Since equivalent representations have the same character, $\chi(g)=\sum_{i=1}^{d_k}R^{(k)}_{ii}(g)$ and $\phi(g)=\sum_{j=1}^{d_l}R^{(l)}_{jj}(g)$ for some k and l.)

Solution.

Let R and S be representations whose characters are χ and ϕ. If $R^{(1)}$, $\mathbb{R}^{(2)}$, ... , $R^{(s)}$ are a full set of irreducible unitary representations of G then R must be equivalent to $R^{(k)}$ and S to $R^{(l)}$ for some k and l. As in the notes, for each $h\in\{1,2,\ldots,s\}$ and p, $m\in\{1,2,\ldots,d_h\}$ define $R^{(h)}_{pm}\colon G\to\mathbb{C}$ by the rule that $R^{(h)}_{pm}g$ is the (p,m)-entry of $R^{(h)}g$ (for each $g\in G$), and let $\chi^{(h)}=\sum_{m=1}^{d_h}R^{(h)}_{mm}$ (the character of $R^{(h)}$). Since equivalent representations have the same character we obtain $\chi=\chi^{(k)}$ and $\phi=\chi^{(l)}$. Now orthogonality of coordinate functions gives

$$\frac{1}{|G|}\sum_{g\in G}(R^{(l)}_{pm}g)\overline{(R^{(k)}_{qn}g)}=(1/d_k)\delta_{kl}\delta_{pq}\delta_{mn},$$

and putting $p=m$ and $q=n$ and summing over m and n gives

$$\frac{1}{|G|}\sum_{g\in G}\Big(\sum_{m=1}^{d_l}(R^{(l)}_{mm}g)\Big)\Big(\sum_{n=1}^{d_k}\overline{(R^{(k)}_{nn}g)}\Big)=(1/d_k)\delta_{kl}\sum_{m=1}^{d_l}\sum_{n=1}^{d_k}\delta_{mn}\delta_{mn}.$$

The right hand side is zero unless $l=k$, in which case it equals 1 (since there are d_k nonzero terms in the sum). So

$$\frac{1}{|G|}\sum_{g\in G}\chi^{(l)}(g)\overline{\chi^{(k)}(g)}=\delta_{kl}. \qquad (*)$$

The left hand side here equals $\sum_{g\in G}\phi(g)\overline{\chi(g)}$. If $\chi\neq\phi$ then certainly $k\neq l$, and so the right hand side is 0. If $\chi=\phi$ the left hand side can be written as $(1/|G|)\sum_{g\in G}|\chi(g)|^2$, which is nonzero since all terms are nonnegative, and $|\chi(1)|^2\neq 0$ as $\chi(1)$ is the trace of the $d_l\times d_l$ identity matrix, which is d_l. So if $\chi=\phi$ we must have $k=l$, and $(*)$ gives $\sum_{g\in G}\phi(g)\overline{\chi(g)}=1$.

2. Use Exercise 1 to show that if χ is the character of a representation which is not irreducible then $(1/|G|)\sum_{g\in G}|\chi(g)|^2>1$.

Solution.

If a representation R is not irreducible then by Maschke's Theorem it is equivalent to the diagonal sum of two other representations, and then its character will be the sum of the characters of these other representations. If these in turn are not irreducible then they can also be written as sums of other characters. As the degrees of the representations are reduced at each step, and the degree of a representation is always a positive integer, the process cannot go on indefinitely. Eventually our original character χ is expressed as a sum $\psi_1+\psi_2+\cdots+\psi_n$, where the ψ_j are characters of irreducible representations. As in Exercise 1, each ψ_j must equal one of the $\chi^{(k)}$ (and there could be repetitions). Thus $\chi=\sum_{k=1}^{s}m_k\chi^{(k)}$ for some nonnegative integers m_k, at least one of which is nonzero. Now

$$\frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2=\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\chi(g)$$

$$=\sum_{h=1}^{s}\sum_{k=1}^{s}\frac{m_hm_k}{|G|}\sum_{g\in G}\overline{\chi^{(h)}(g)}\chi^{(k)}(g)=\sum_{h=1}^{s}\sum_{k=1}^{s}m_hm_k\delta_{hk}=\sum_{k=1}^{s}m_k^2.$$

The least value this can take is 1. Moreover, this minimum is only attained when one of the m_k's is 1 and the others all 0, in which case $\chi=\chi^{(k)}$ is irreducible. Otherwise $(1/|G|)\sum_{g\in G}|\chi(g)|^2>1$.

3. Prove that if λ is a character of G of degree 1 and ϕ is any character of G then $\lambda\phi$, defined by $(\lambda\phi)(g)=\lambda(g)\phi(g)$ for all $g\in G$, is also a character of G.

Solution.

Let R be a matrix representation with character ϕ. (Note that λ is a representation as well as a character, since its degree is 1.) Define $S(g) = \lambda(g)R(g)$ for all $g \in G$. Then S is a representation, since for all g and h in G

$$\begin{aligned} S(gh) = \lambda(gh)R(gh) &= \lambda(g)\lambda(h)R(g)R(h) \\ &= \lambda(g)R(g)\lambda(h)R(h) = S(g)S(h) \end{aligned}$$

(since scalars commute with matrices). Multiplying a matrix by a scalar clearly multiplies its trace by the same scalar. The trace of $R(g)$ is $\chi(g)$; hence the trace of $S(g)$ is $\lambda(g)\chi(g) = (\lambda\chi)(g)$, and the function $g \mapsto (\lambda\phi)(g)$ is the character of the representation S.

4. Determine the irreducible characters of S_4, given that there are exactly five of them.

Solution.

Each element of S_4 is uniquely expressible in the form σx with $\sigma \in S_3$ and $x \in K$, and each coset of K in S_4 is uniquely expressible as σK with $\sigma \in S_3$. A character χ of S_3 becomes a character of S_4/K if we define $\chi(\sigma K) = \chi(\sigma)$ for all $\sigma \in S_3$. By Exercise 2 we obtain a character $\tilde{\chi}$ of S_4 satisfying $\tilde{\chi}(\sigma x) = \chi(\sigma K) = \chi(\sigma)$ for all $\sigma \in S_3$ and all $x \in K$. We see that $\tilde{\chi}$ must take the same value on elements of K as it takes at the identity, and the same value on four-cycles as on transpositions. (Observe, for instance, that $(1,2,3,4) = (1,3)[(2,3)(1,4)]$.) Starting with the 1-character of S_3 this simply gives the 1-character of S_4, and similarly the sign character of S_3 (mapping even permutations to 1 and odd permutations to -1) gives the sign character of S_4. The irreducible character of S_3 of degree 2 yields χ_3 of the table below. The character χ_4 was found in Assignment 1, and χ_5 is the product of χ_4 and χ_2 (see Exercise 1).

	1	$(1,2)$	$(1,2,3)$	$(1,2,3,4)$	$(1,2)(3,4)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

The numbers of elements in the various conjugacy classes are 1, 6, 8, 6 and 3 (taking the classes in the same order as in the table). To calculate the inner product of a character with itself, multiply the square of the absolute value of the character on each class by the number of elements in the class, sum over all the classes and divide by the order of the group (24 in this case). Thus for χ_4 we get

$$(3^2 + 6 \times 1^2 + 8 \times 0^2 + 6 \times (-1)^2 + 3 \times (-1)^2)/24$$

which is 1. Since each of the χ_i give 1 they are all irreducible.

5. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{kl})$ a $p \times q$ matrix then the *Kronecker product* of A and B is the $mp \times nq$ matrix $A \dot{\times} B$ whose $((i-1)p+k, (j-1)q+l)$-entry is $a_{ij}b_{kl}$. That is,

$$A \dot{\times} B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}.$$

Prove that $(A \dot{\times} B)(C \dot{\times} D) = AC \dot{\times} BD$, provided that the number of rows of C (resp. D) equals the number of columns of A (resp. B).

Solution.

By multiplication of partitioned matrices in the usual way we find that $(A \dot{\times} B)(C \dot{\times} D)$ equals

$$\begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} \begin{pmatrix} c_{11}D & c_{12}D & \dots & c_{1r}D \\ c_{21}D & c_{22}D & \dots & c_{2r}D \\ \vdots & \vdots & & \vdots \\ c_{n1}D & c_{n2}D & \dots & c_{nr}D \end{pmatrix}$$

$$= \begin{pmatrix} \sum_j a_{1j}c_{j1}BD & \sum_j a_{1j}c_{j2}BD & \dots & \sum_j a_{1j}c_{jr}BD \\ \sum_j a_{2j}c_{j1}BD & \sum_j a_{2j}c_{j2}BD & \dots & \sum_j a_{2j}c_{jr}BD \\ \vdots & \vdots & & \vdots \\ \sum_j a_{mj}c_{j1}BD & \sum_j a_{mj}c_{j2}BD & \dots & \sum_j a_{mj}c_{jr}BD \end{pmatrix},$$

which equals $AC \dot{\times} BD$, since $\sum_j a_{ij}c_{jk}$ is the (i,k)-entry of AC.