Fusion procedure for the symmetric group

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GL07, July 2007

The symmetric group \mathfrak{S}_k acts naturally on the tensor product space

$$\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N$$
, *k* factors,

by permuting the factors. On the other hand, \mathbb{C}^N carries the vector representation of the Lie algebra \mathfrak{gl}_N so that the tensor product space is a representation of \mathfrak{gl}_N .

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The actions of \mathfrak{S}_k and \mathfrak{gl}_N commute with each other.

Irreducible decomposition of the \mathfrak{gl}_N -module

$$(\mathbb{C}^N)^{\otimes k} \cong \bigoplus_{\lambda} f_{\lambda} L(\lambda),$$

where λ runs over partitions $\lambda = (\lambda_1, \dots, \lambda_N)$,

 $\lambda_1 \ge \cdots \ge \lambda_N \ge 0$ such that $\lambda_1 + \cdots + \lambda_N = k$,

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 f_{λ} is the dimension of the irreducible representation of \mathfrak{S}_k associated with λ .

 f_{λ} equals the number of standard λ -tableaux \mathcal{U} .

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Refined decomposition

$$(\mathbb{C}^{N})^{\otimes k} \cong \bigoplus_{\lambda \vdash k} \bigoplus_{\mathsf{sh}(\mathcal{U}) = \lambda} \Phi_{\mathcal{U}}(\mathbb{C}^{N})^{\otimes k},$$

where each subspace $L_{\mathcal{U}} = \Phi_{\mathcal{U}}(\mathbb{C}^N)^{\otimes k}$

is a \mathfrak{gl}_N -submodule isomorphic to $L(\lambda)$.

If $\mathcal{U} = \mathcal{U}^r$ is the row tableau of shape λ , then the subspace $L_{\mathcal{U}^r}$ coincides with the image of the Young symmetrizer,

$$L_{\mathcal{U}^r} = H_{\mathcal{U}^r} A_{\mathcal{U}^r} (\mathbb{C}^N)^{\otimes k},$$

where $H_{\mathcal{U}^r}$ and $A_{\mathcal{U}^r}$ are the row symmetrizer and column anti-symmetrizer of \mathcal{U}^r .

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Problem: Find an explicit formula for the element

$$\phi_{\mathcal{U}} \in \mathbb{C}[\mathfrak{S}_k]$$

whose image in the representation of \mathfrak{S}_k coincides with $\Phi_{\mathcal{U}}$.



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Young basis

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The orthonormal Young basis $\{v_{\mathcal{U}}\}$ of V_{λ} is parameterized by the set of standard λ -tableaux \mathcal{U} .

For any $i \in \{1, ..., k - 1\}$ set $s_i = (i, i + 1)$. We have

$$s_i \cdot v_{\mathcal{U}} = d v_{\mathcal{U}} + \sqrt{1 - d^2} v_{s_i \mathcal{U}},$$

where $d = (c_{i+1} - c_i)^{-1}$, $c_i = c_i(\mathcal{U})$ is the content b - aof the cell (a, b) occupied by *i* in a standard λ -tableau \mathcal{U} , and the tableau $s_i\mathcal{U}$ is obtained from \mathcal{U} by swapping the entries *i* and i + 1. The group algebra $\mathbb{C}[\mathfrak{S}_k]$ is isomorphic to the direct sum of matrix algebras

$$\mathbb{C}[\mathfrak{S}_k] \cong \bigoplus_{\lambda \vdash k} \operatorname{Mat}_{f_{\lambda}}(\mathbb{C}),$$

where $f_{\lambda} = \dim V_{\lambda}$. The matrix units $e_{\mathcal{UU}'} \in \operatorname{Mat}_{f_{\lambda}}(\mathbb{C})$ are parameterized by pairs of standard λ -tableaux \mathcal{U} and \mathcal{U}' .

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$$e_{\mathcal{U}\mathcal{U}'} = rac{f_{\lambda}}{k!} \phi_{\mathcal{U}\mathcal{U}'}$$

where $\phi_{UU'}$ is the matrix element corresponding to the basis vectors v_{U} and $v_{U'}$ of the representation V_{λ} ,

$$\phi_{\mathcal{U}\mathcal{U}'} = \sum_{\boldsymbol{s}\in\mathfrak{S}_k} (\boldsymbol{s}\cdot\boldsymbol{v}_{\mathcal{U}},\boldsymbol{v}_{\mathcal{U}'})\cdot\boldsymbol{s}^{-1}\in\mathbb{C}[\mathfrak{S}_k].$$

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$$1 = \sum_{\lambda \vdash k} \sum_{\mathsf{sh}(\mathcal{U}) = \lambda} e_{\mathcal{U}},$$

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the elements we want are $\phi_{\mathcal{U}}$ (or $e_{\mathcal{U}}$), yielding

$$(\mathbb{C}^N)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} \bigoplus_{\mathsf{sh}(\mathcal{U}) = \lambda} \Phi_{\mathcal{U}}(\mathbb{C}^N)^{\otimes k}$$

The Jucys–Murphy elements of $\mathbb{C}[\mathfrak{S}_k]$ are defined by

$$x_1 = 0,$$
 $x_i = (1 i) + (2 i) + \dots + (i - 1 i),$ $i = 2, \dots, k.$

They generate a commutative subalgebra of $\mathbb{C}[\mathfrak{S}_k]$. Moreover, x_k commutes with all elements of \mathfrak{S}_{k-1} .

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The vectors of the Young basis are eigenvectors for the action of x_i on V_{λ} . For any standard λ -tableau \mathcal{U} we have

$$x_i \cdot v_{\mathcal{U}} = c_i(\mathcal{U}) v_{\mathcal{U}}, \qquad i = 1, \ldots, k.$$

The branching properties of the Young basis imply the corresponding properties of the matrix units. If \mathcal{V} is a given standard tableau with the entries $1, \ldots, k-1$ then

$$oldsymbol{e}_{\mathcal{V}} = \sum_{\mathcal{V}
ightarrow \mathcal{U}} oldsymbol{e}_{\mathcal{U}},$$

where $\mathcal{V} \to \mathcal{U}$ means that the standard tableau \mathcal{U} is obtained from \mathcal{V} by adding one cell with the entry *k*.

Furthermore,

$$x_i e_{\mathcal{U}} = e_{\mathcal{U}} x_i = c_i(\mathcal{U}) e_{\mathcal{U}}, \qquad i = 1, \dots, k$$

for any standard λ -tableau \mathcal{U} ,

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and we have the identity in $\mathbb{C}[\mathfrak{S}_k]$,

$$x_k = \sum_{\lambda \vdash k} \sum_{\operatorname{sh}(\mathcal{U}) = \lambda} c_k(\mathcal{U}) e_{\mathcal{U}},$$

so that x_k can be viewed as a diagonal matrix.

Now let $k \ge 2$ and let λ be a partition of k. Fix a standard λ -tableau \mathcal{U} and denote by \mathcal{V} the standard tableau obtained from \mathcal{U} by removing the cell α occupied by k. Denote the shape of \mathcal{V} by μ .

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Murphy's formula. We have the relation in $\mathbb{C}[\mathfrak{S}_k]$,

$$\boldsymbol{e}_{\mathcal{U}} = \boldsymbol{e}_{\mathcal{V}} \, \frac{(x_k - a_1) \dots (x_k - a_l)}{(c - a_1) \dots (c - a_l)},$$

where a_1, \ldots, a_l are the contents of all addable cells of μ except for α , while *c* is the content of the latter.

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Equivalently,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{u-c}{u-x_k}\Big|_{u=c}.$$

Proof.

Write

$$oldsymbol{e}_{\mathcal{V}} = \sum_{\mathcal{V}
ightarrow \mathcal{U}'} oldsymbol{e}_{\mathcal{U}'}.$$

Then $x_k e_{\mathcal{U}'} = a_i e_{\mathcal{U}'}$ for some *i* if $\mathcal{U}' \neq \mathcal{U}$ while $x_k e_{\mathcal{U}} = c e_{\mathcal{U}}$.

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Then $x_k e_{\mathcal{U}'} = a_i e_{\mathcal{U}'}$ for some *i* if $\mathcal{U}' \neq \mathcal{U}$ while $x_k e_{\mathcal{U}} = c e_{\mathcal{U}}$.

Similarly,

$$e_{\mathcal{V}} \frac{u-c}{u-x_k} = \sum_{\mathcal{V} \to \mathcal{U}'} e_{\mathcal{U}'} \frac{u-c}{u-c_k(\mathcal{U}')} = e_{\mathcal{U}} + \sum_{\mathcal{V} \to \mathcal{U}', \mathcal{U}' \neq \mathcal{U}} e_{\mathcal{U}'} \frac{u-c}{u-c_k(\mathcal{U}')}.$$

Since $c_k(\mathcal{U}') \neq c$ for all standard tableaux \mathcal{U}' distinct from \mathcal{U} , the value of this rational function at u = c is $e_{\mathcal{U}}$.

Corollary

We have

$$\phi_{\mathcal{U}} = \mathcal{H}_{\lambda,\mu} \, \phi_{\mathcal{V}} \, \frac{u-c}{u-x_k} \Big|_{u=c}$$

with

$$\mathcal{H}_{\lambda,\mu}=rac{(a_1-c)\ldots(a_p-c)(c-a_{p+1})\ldots(c-a_l)}{(b_1-c)\ldots(b_q-c)(c-b_{q+1})\ldots(c-b_r)},$$

where the numbers $a_1, \ldots, a_p, c, a_{p+1}, \ldots, a_l$ are the contents of all addable cells of μ and $b_1, \ldots, b_q, c, b_{q+1}, \ldots, b_r$ are the contents of all removable cells of λ with both sequences written in the decreasing order.

Remark

Consider the character χ_{λ} of V_{λ} ,

$$\chi_{\lambda} = \sum_{\boldsymbol{s} \in \mathfrak{S}_k} \chi_{\lambda}(\boldsymbol{s}) \, \boldsymbol{s} \in \mathbb{C}[\mathfrak{S}_k].$$

We have

$$\chi_{\lambda} = \sum_{\mathsf{sh}(\mathcal{U}) = \lambda} \phi_{\mathcal{U}},$$

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summed over all standard λ -tableaux \mathcal{U} .

Hence for the normalized characters $\widehat{\chi}_{\lambda} = f_{\lambda} \chi_{\lambda} / k!$ we have

$$\widehat{\chi}_{\lambda} = \sum_{\mu \to \lambda} \widehat{\chi}_{\mu} \frac{(x_k - a_1) \dots (x_k - a_l)}{(c - a_1) \dots (c - a_l)}.$$

For any distinct indices $i, j \in \{1, ..., k\}$ introduce the rational function in two variables u, v with values in the group algebra $\mathbb{C}[\mathfrak{S}_k]$ by

$$\rho_{ij}(u,v) = 1 - \frac{(ij)}{u-v}.$$

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Take *k* complex variables u_1, \ldots, u_k and set

$$\phi(u_1,\ldots,u_k) = \rho_{12}(u_1,u_2) \rho_{13}(u_1,u_3) \rho_{23}(u_2,u_3)$$
$$\times \ldots \rho_{1k}(u_1,u_k) \rho_{2k}(u_2,u_k) \ldots \rho_{k-1,k}(u_{k-1},u_k).$$

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Motivation: The image of $\rho_{ij}(u, v)$ in End $(\mathbb{C}^N)^{\otimes k}$ is the Yang *R*-matrix.

Theorem

Suppose that λ is a partition of k and let $\mathcal U$ be a standard

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of the rational function $\phi(u_1, ..., u_k)$ are well-defined. The corresponding value coincides with the matrix element $\phi_{\mathcal{U}}$,

$$\phi_{\mathcal{U}} = \phi(u_1, \ldots, u_k) \big|_{u_1 = c_1} \big|_{u_2 = c_2} \cdots \big|_{u_k = c_k}.$$

Example: $\lambda = (k)$. Then

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is the symmetrizer in $\mathbb{C}[\mathfrak{S}_k]$.

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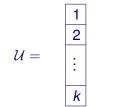
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is the symmetrizer in $\mathbb{C}[\mathfrak{S}_k]$. By the theorem,

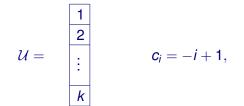
$$\phi_{\mathcal{U}} = \left(1 + \frac{(1\,2)}{1}\right) \left(1 + \frac{(1\,3)}{2}\right) \left(1 + \frac{(2\,3)}{1}\right)$$
$$\times \dots \left(1 + \frac{(1\,k)}{k-1}\right) \left(1 + \frac{(2\,k)}{k-2}\right) \dots \left(1 + \frac{(k-1\,k)}{1}\right).$$

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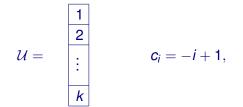
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and $\phi_{\mathcal{U}} = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot \sigma$ is the anti-symmetrizer in $\mathbb{C}[\mathfrak{S}_k]$,

$$\phi_{\mathcal{U}} = \left(1 - \frac{(1\,2)}{1}\right) \left(1 - \frac{(1\,3)}{2}\right) \left(1 - \frac{(2\,3)}{1}\right)$$
$$\times \dots \left(1 - \frac{(1\,k)}{k-1}\right) \left(1 - \frac{(2\,k)}{k-2}\right) \dots \left(1 - \frac{(k-1\,k)}{1}\right).$$

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$$\phi_{\mathcal{U}} = \left(1 + (12)\right) \left(1 - (13)\right) \left(1 - \frac{(23)}{2}\right),$$

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while $c_1 = 0$, $c_2 = -1$, $c_3 = 1$ for \mathcal{V} , and

$$\phi_{\mathcal{V}} = \left(1 - (12)\right) \left(1 + (13)\right) \left(1 + \frac{(23)}{2}\right).$$

Example: $\lambda = (2^2)$,

 $\phi(u_1, u_2, u_3, u_4) = \rho_{12}(u_1, u_2) \rho_{13}(u_1, u_3) \rho_{23}(u_2, u_3)$ $\times \rho_{14}(u_1, u_4) \rho_{24}(u_2, u_4) \rho_{34}(u_3, u_4).$

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Take the standard λ -tableau

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The contents are $c_1 = 0$, $c_2 = 1$, $c_3 = -1$, $c_4 = 0$.

Taking
$$u_1 = 0$$
, $u_2 = 1$, $u_3 = -1$, $u_4 = u$ we get

$$\phi(0, 1, -1, u) = \left(1 + (12)\right) \left(1 - (13)\right) \left(1 - \frac{(23)}{2}\right)$$
$$\times \left(1 + \frac{(14)}{u}\right) \left(1 + \frac{(24)}{u-1}\right) \left(1 + \frac{(34)}{u+1}\right).$$

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By the theorem, this rational function is regular at u = 0 and the corresponding value coincides with ϕ_{12} .

We have

$$\phi(0,1,-1,u) = \phi_{\mathcal{V}}\left(1 + \frac{(1\,4)}{u}\right)\left(1 + \frac{(2\,4)}{u-1}\right)\left(1 + \frac{(3\,4)}{u+1}\right),$$

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where

$$\mathcal{V} = \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}$$

Next step:

$$\phi_{\mathcal{V}}\left(1+\frac{(1\,4)}{u}\right)\left(1+\frac{(2\,4)}{u-1}\right)\left(1+\frac{(3\,4)}{u+1}\right)$$
$$=\prod_{i=1}^{3}\left(1-\frac{1}{(u-c_{i})^{2}}\right)\frac{u}{u-c_{4}}\cdot\phi_{\mathcal{V}}\frac{u-c_{4}}{u-x_{4}},$$

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where $c_1 = 0, c_2 = 1, c_3 = -1, c_4 = 0$ and

 $x_4 = (14) + (24) + (34).$

Finally, apply Murphy's formula to get

$$\prod_{i=1}^{3} \left(1 - \frac{1}{(u-c_i)^2}\right) \frac{u}{u-c_4} \cdot \phi_{\mathcal{V}} \frac{u-c_4}{u-x_4}\Big|_{u=c_4} = \phi_{\mathcal{U}}.$$

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Thus,

$$\begin{split} \phi_{\mathcal{U}} &= \phi(0, 1, -1, 0) \\ &= \frac{1}{2} \left(1 + (12) \right) \left(1 - (13) \right) \left(2 - (23) \right) \\ &\times \left(2 - (14) - (24) - (34) \right) \left(2 + (14) + (24) + (34) \right). \end{split}$$