THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH3063

DIFFERENTIAL EQUATIONS & BIOMATHS (N)

Semester 1
June 2008
Time Allowed: Two Hours
Lecturer: A. Nelson

This exam consists of 4 pages numbered 1 to 4.
There are 4 questions, numbered 1 to 4.

All 4 questions may be attempted.
Reasons and working must be given.

Each question is marked out of 25. Part marks as shown.

The following notation is assumed,

\[ \dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}. \]
1. (a) Consider the linear system
\[ \dot{x} = x - 2y, \quad \dot{y} = -2x + y. \]

(i) Determine the eigenvalues and corresponding eigenvectors for coefficient matrix of this system. [5 marks]

(ii) Sketch the phase portrait of the system.
Include in your diagram the systems nullclines and normal modes. [6 marks]

(iii) This is a gradient system. Find the gradient function \( F(x, y) \) for the system which satisfies \( F(0, 0) = 0 \). [3 marks]

(b) What is a limit cycle? Make a sketch to illustrate a stable (attracting) limit cycle. [3 marks]

(c) For all \( \mu \) the point \((0, 0)\) is an equilibrium of the system
\[ \dot{x} = \mu x + y - x^3, \quad \dot{y} = -x + \mu y - y^3. \]

(i) Show that this family of systems undergoes a Hopf bifurcation at \( \mu = 0 \). [4 marks]

(ii) Given that for \( \mu = 0 \), \((0, 0)\) is sink, describe what happens at this Hopf bifurcation as \( \mu \) goes from negative to positive. [4 marks]

2. (a) Consider the conservative system
\[ \dot{x} = y, \quad \dot{y} = x(1 - x), \]
equivalent to the second order equation
\[ \ddot{x} = x(1 - x). \]

(i) Do phase plane analysis of this system, i.e. sketch and label the horizontal and vertical nullclines, indicate equilibrium points, and label regions to show the signs of \( \dot{x} \) and \( \dot{y} \). [8 marks]

(ii) Verify that
\[ H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} \]
is a Hamiltonian function \( H(x, y) \) for this system. [2 marks]

(iii) Do a linear analysis of the equilibrium points of the system.
What conclusions can you draw in this case? [6 marks]

(iv) Make a sketch of the phase portrait of this system. [4 marks]

(b) Show that \( V(x, y) = 2x^2 + 3y^2 \) is a strict Liapunov function for the equilibrium point \((0, 0)\) of the system
\[ \dot{x} = -3y - x^3, \quad \dot{y} = 2x - y^3. \]

What can you conclude? [5 marks]

\[ \text{i.e.} \quad V < 0 \text{ except at } (0,0) \]

\[ \text{like } V = 0 \]

EXAM CONTINUES
3. The growth of a fish population is modelled by the differential equation

$$\dot{x} = f(x) = rx(x/m - 1)(1 - x/M),$$

where $x$ is population as a function of time $t$, and $r > 0$ and $M > m > 0$ are constants.

(a) (i) Sketch a representative graph of $\dot{x} = f(x)$ as a function of $x$.

Hence, or otherwise, sketch the phase line of the equation, indicating the equilibrium populations and their nature. [4 marks]

(ii) Give a population model theoretic interpretation of the constants $m$ and $M$. [2 marks]

(b) Consider now harvesting the population by a constant effort strategy.

Suppose this is modelled by the harvest equation,

$$\dot{x} = f(x) - Ex = rx(x/m - 1)(1 - x/M) - Ex,$$

where the control parameter $E \geq 0$ is a measure of the effort.

(i) Note for all $E$, $x = 0$ is an equilibrium population.

Show this equilibrium, $x = 0$, is a sink, for all $E \geq 0$. [3 marks]

(ii) Make a sketch of $\dot{x} = f(x)$ and lines $\dot{x} = Ex$ to show that for some value $E_c > 0$ the harvest equation has one equilibrium point $x_e$ besides $x = 0$, that for $0 \leq E < E_c$ it has two more, $x_1(E)$ and $x_2(E)$, say, and for $E > E_c$ it has no more. [4 marks]

(iii) Sketch the bifurcation diagram of the harvested equation, i.e. plot equilibrium points as a function of $E$, with stable (sinks) and unstable (sources) branches clearly indicated. Identify any bifurcation points. [4 marks]

(iv) Make a sketch of the line $\dot{x} = Ex$, through the maximum turning point of the graph of $f(x)$.

Let $E = E_m$ be the slope of this line.

Explain why $x_2(E_m)$ gives the maximal sustainable yield. [4 marks]

(v) Suppose a previously un-fished population is to be harvested. The fishing authority experiments with incrementally increasing the fishing effort from 0, leaving some time between increments for the population to adjust to the new situation. Discuss how the population and yield change as this experiment is carried out. [4 marks]
4. Consider the following predator prey model, with positive model parameters $r$, $s$, $K$, $a$ and $b$,

$$\frac{dP}{dt} = r \left[ P(1 - P/K) - aPQ \right], \quad \frac{dQ}{dt} = sQ(1 - Q/bP),$$

for prey population $P > 0$, and predator population $Q \geq 0$, functions of time $t$.

*Note the restriction $P > 0$. The $Q$-axis, $P = 0$, is not a vertical nullcline of this system. For $Q > 0$, $dQ/dt \to -\infty$ as $P \to 0^+$.  

(a) State briefly the assumptions of this model [4 marks]

(b) Sketch the nullclines of the system to show that the system has exactly one equilibrium $(\bar{P}, \bar{Q})$ with both species present, and one other $(K, 0)$ with the predator population absent. [3 marks]

(c) Make a phase plane analysis of this system. [4 marks]

(d) Show a linear analysis determines the type, saddle, sink or source, of the two equilibrium points of the system. [5 marks]

(e) Make a sketch and explain why any rectangle in the quadrant $P \geq 0$, $Q \geq 0$ with sides parallel to the coordinate axes, one corner at $(0, 0)$ and the opposite corner at a point $C = (A, B)$ on the line $Q = bP$, with $A \geq K$, traps trajectories. [5 marks]

(f) What does this model predict? [4 marks]