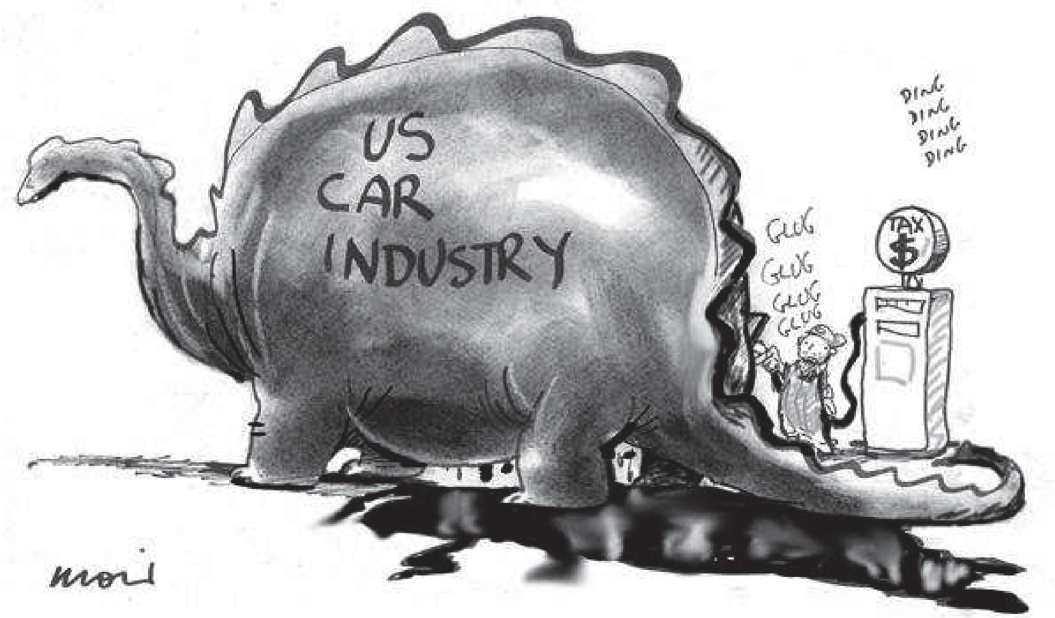
Chapter 5

Resource-Limited Growth



Cartoon reproduced with the permission of the artist, Alan Moir.

> *This chapter uses ideas and techniques from earlier chapters to investigate important applications in economics and the life sciences.*
>
> *In this part of the course you will develop the following generic skill:*
>
> - *Combining many simple tasks and skills to complete a complex task.*
>
> There are no new mathematical techniques introduced in this chapter.

All three applications in this chapter are related to the logistic model.[1]

[1]The word logistics refers to the management of resources: money, goods, ammunition, people or information and the word was originally used by the ancient Greeks in a military context.

5.1 Peak Oil

In 1956 a geophysicist named Marion King Hubbert working at a Shell research lab in Texas analysed the historical trends in US productions of coal, crude oil and natural gas. He noticed

> *during the initial stages all of these rates of production tend to increase exponentially*

In particular, he also calculated that

> *Crude oil production from 1880 until 1930 increased at the rate of 7.9% per year.*

Why should oil production increase exponentially? The basic assumptions of his theory were:

- The *rate* of oil production depends on how many oil wells you can dig.
- The number of oil wells you can dig depends on how much money you have.
- How much money you have depends on how much oil you have already produced and sold.

Thus, Hubbert argued that oil production rates should be proportional to the total amount of oil already produced. If the total amount produced by time t is $P(t)$, then the production rate is $\frac{dP}{dt}$ and Hubbert constructed the model

$$\frac{dP}{dt} = rP. \tag{5.1}$$

We know that this differential equation corresponds to exponential growth and the historical data revealed that the relative growth rate r was 1.079 per year for the Texan oil fields. However, Hubbert also realised that

> *No finite resource can sustain for longer than a brief period such a rate of growth of production; therefore, although production rates tend initially to increase exponentially, physical limits prevent their continuing to do so.*

He decided to use a logistic model to include the effects of limited resources. He modified the equation to

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}). \tag{5.2}$$

The extra factor $1 - \frac{P}{K}$ takes into account how much more difficult it becomes to extract oil as the oil begins to run out. The quantity K is called the capacity and represents the total amount of oil that was in the oil-field to start with. As P approaches K this extra factor becomes smaller and smaller.

Using separation of variables we can write

$$\int \frac{K}{r} \frac{dP}{P(K-P)} = \int dt \tag{5.3}$$

Then using partial fractions and integration we can obtain:

$$\frac{1}{r}[\ln|P| - \ln|K-P|] = t + C \tag{5.4}$$

where C is an arbitrary constant of integration. Let T be the time at which half the total oil reserves $P = \frac{1}{2}K$ have been extracted. The general solution can then be rewritten as

$$\frac{1}{r}[\ln|P| - \ln|K-P|] = t - T \tag{5.5}$$

Just as for an epidemic, the peak of oil production occurs when exactly half the oil reserve has been extracted.

We are only interested in situations where $0 < P < K$ so we can get rid of the absolute value signs and make P the subject. This gives

$$\boxed{P(t) = K\frac{e^{r(t-T)}}{1+e^{r(t-T)}}} \tag{5.6}$$

This is our good old friend, the logistic function, with an extra factor of K out the front.

If we substitute this function back into the RHS of the original differential equation we can get a formula for the derivative of the logistic function:

$$\frac{dP}{dt} = r\,P(1-\frac{P}{K}) = rK\frac{e^{r(t-T)}}{(1+e^{r(t-T)})^2} \tag{5.7}$$

A plot of this derivative (which is the growth rate) is shown in Fig. 5.1. As we expect, the growth is slow at first, peaks in the middle, and then slows again at the end. The peak occurs when P is half the total capacity K. This occurs when $t = T$ and at this point the formula above predicts that the absolute growth rate is $\frac{1}{4}rK$.

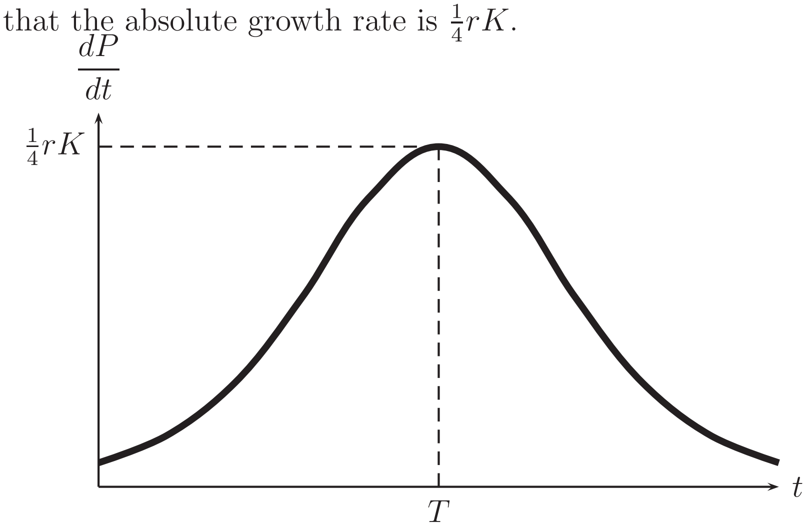


Figure 5.1: The absolute growth rate is the derivative of the logistic function. Notice the peak at $t = T$. The curve is symmetric about this point.

In the context of natural resource management, the curve described by Eq. (5.7) and Fig. 5.1 is called the Hubbert curve, and its peak is called the Hubbert peak.

Hubbert's most optimistic estimate of the total Texan oil reserves was $K = 200$ gigabarrels. Based on these parameters he predicted that Texan oil production would peak in 1970. The agreement between his predictions and the actual production rates can be seen in Fig. 5.2. This is a surprisingly good result given the simplicity of the underlying mathematical theory. A similar analysis of Norwegian oil production is shown in Fig. 5.3.

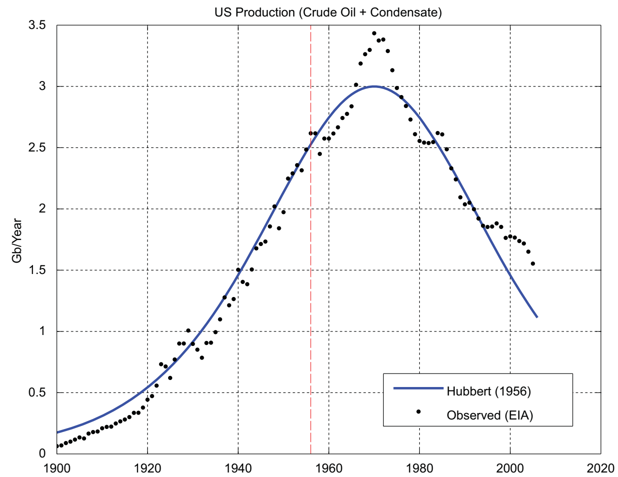


Figure 5.2: US oil production (1900-2000). Reproduced from wikipedia under the Wikimedia Commons license.

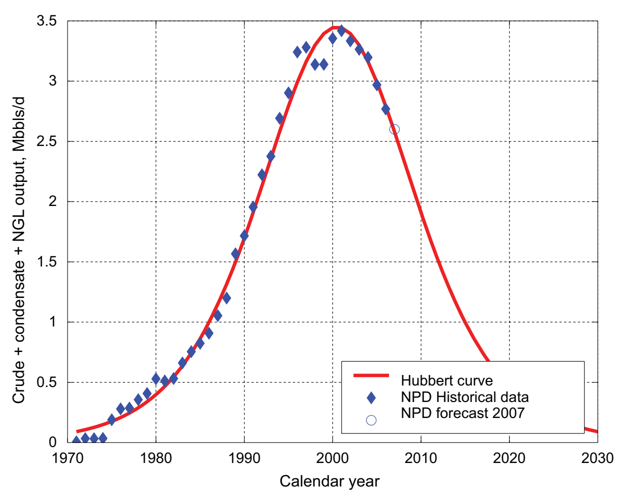
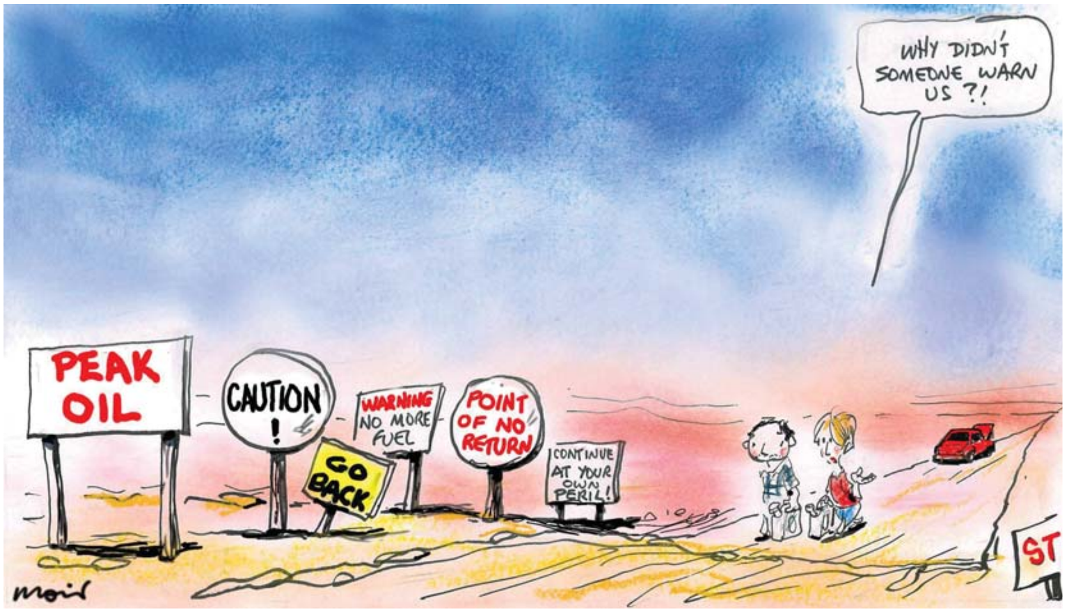


Figure 5.3: Norwegian oil production (1970-2005). Reproduced from wikipedia under the Wikimedia Commons license.



Cartoon reproduced with the permission of the artist, Alan Moir.

5.2 Competition within species

When resources (such as food, water or space) are limited, members of the same species will end up competing with each other for these resources. Under such conditions the *relative growth rate* is no longer constant but depends on the size or density of the population.

The simplest such mathematical model was developed by Pierre Francois Verhulst who published it in 1838:

$$\frac{1}{P}\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right). \tag{5.8}$$

The LHS is the relative growth rate. On the RHS when the population is small, the relative growth rate is r. This is called the *intrinsic growth rate.* As the population increases, the RHS gets smaller. When P reaches K, the population stops growing. The value K is called the carrying capacity. When the population is larger than the carrying capacity, the growth rate becomes negative and the population begins to shrink. If we rearrange the above equation

$$\frac{dP}{dt} = r\,P\left(1 - \frac{P}{K}\right). \tag{5.9}$$

we see that it is the by now familiar logistic equation. The model has an unstable equilibrium at zero and a stable equilibrium when $P = K$. The population is growing at its fastest when the population is at half the carrying capacity.

In Biology the logistic model is often called the Verhulst model.

Example:

Assume that the size of a population obeying the Verhulst model is initially twice the carrying capacity. Determine $P(t)$.

Solution:

The calculation to determine $P(t)$ by separation of variables and partial fractions is exactly the same as in the previous section until we get to the following equation

$$\frac{1}{r}\left[\ln|P| - \ln|K - P|\right] = t + C. \tag{5.10}$$

We can remove the absolute value signs if we remember that in this situation $P > K$, so we should write

$$\frac{1}{r}\left[\ln(P) - \ln(P-K)\right] = t + C. \tag{5.11}$$

When $t = 0$ we have that $P = 2K$ thus we can determine that

$$\frac{1}{r}\left[\ln(2K) - \ln(K)\right] = C \tag{5.12}$$

and after rearranging

$$C = \frac{\ln(2)}{r}. \tag{5.13}$$

Thus, if we substitute this into the general solution and collect all the logarithms on one side we can see that

$$\frac{1}{r}\ln\left(\frac{P}{2(P-K)}\right) = t. \tag{5.14}$$

If we now make try to make P the subject we get

$$\frac{P}{2(P-K)} = e^{rt} \tag{5.15}$$

and then

$$P = K\frac{2e^{rt}}{2e^{rt}-1}. \tag{5.16}$$

5.2.1 r-K Selection Theory

One knows that a mathematical equation has had a tremendous impact on a field when scientists start using the names of mathematical variables to describe their own non-mathematical theories. There are two parameters in the Verhulst or logistic model: r and K. In turns out that biological species have evolved to exploit the advantages of either

- trying to make r as large as possible (i.e. reproduce a lot) or,
- trying to make K as large as possible (i.e. work out how best to exploit limited resources).

The terms r-selection and K-selection were coined by Robert MacArthur and Edwin Wilson in 1967. The basic idea is that these two extreme types of species can be recognised by their behaviour.

The r-selected species (also called opportunistic species) do well in unstable environments and try to quickly exploit new niches, have lots of offspring, but each offspring has only a small probability of survival. They tend to be small, short lifespan species like bacteria, most insects and small plants.

The K-selected species (also called equilibrium species) do well in very stable environments and are well adapted to exploiting existing crowded niches, have few offspring, but each offspring has a high probability of survival. They tend to be large, long lifespan species like elephants, whales and trees.

5.3 Sustainable Harvesting

The logistic equation can be modified to model the effects of fishing on fish populations and compare different models of harvesting. Thus we will look at modifications to the logistic model that incorporate the effect of humans by subtracting something from the growth rate

$$\frac{dP}{dt} = r\,P\left(1 - \frac{P}{K}\right) - \text{human effects.} \tag{5.17}$$

There are two important models.

5.3.1 Constant Harvest Model

Consider the following model

$$\frac{dP}{dt} = r\,P\left(1 - \frac{P}{K}\right) - h. \tag{5.18}$$

The first term on the RHS is the usual logistic model, the second term represents depletion due to fishing which is assumed to occur at a *constant absolute rate* given by h.

A constant absolute harvest rate would occur in industries that impose a fixed quota (for example, by counting the number of fish that are caught).

The right-hand-side is still a quadratic function of P but the parabola has now been shifted downwards by a distance h.

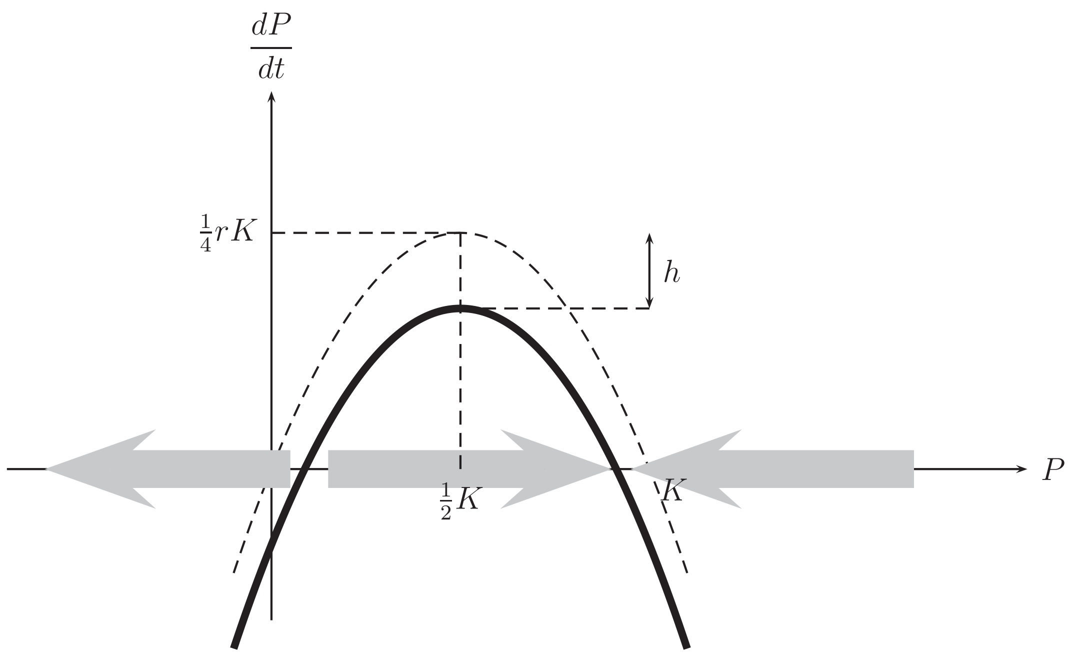


Figure 5.4: Stable and unstable equilibria for the constant harvest model.

We can see from Fig. 5.4 that if the distance h is not too large there are still two equilibria, but they are now closer together. These are given by the two solutions of the quadratic equation

$$r\,P\left(1 - \frac{P}{K}\right) - h = 0. \tag{5.19}$$

The smaller equilibrium is unstable, but the larger equilibrium is stable.

However, if the harvesting rate h is too large then the parabola will be shifted below the axis and there will be no equilibrium. There are two ways to work out the maximum harvesting rate.

The first method is by using the diagram. The peak of the original parabola occurs at $P = \frac{1}{2}K$ and the growth rate at this point is $\frac{1}{4}rK$. Thus, this is also the largest distance that the parabola can be shifted downwards and still cross the axis. In many economics and biology textbooks this the maximum rate is called the *Maximum Sustainable Yield*:

$$MSY = h_{max} = \frac{1}{4}rK \tag{5.20}$$

The second method is by analysing quadratic equations. A quadratic equation has solutions if the discriminant is not negative. For Eq. (5.19) this discriminant condition gives

$$r^2 - 4\frac{r}{K}h \geq 0 \tag{5.21}$$

which is equivalent to

$$h \leq h_{max} = \frac{1}{4}rK. \tag{5.22}$$

Fisheries management models based on MSY were very popular in the 1950s and were even part of various treaties and United Nations conventions in the 80s.

5.3.2 Constant Effort Model

A second very important model is the Schaefer Short-Term Catch equation which was developed by Milner Bailey Schaefer while he was director of the Inter-American Tropical Tuna Commission. It assumes that fish are caught with a constant *effort* e and that the amount of fish caught is given by multiplying the effort by the fish population to give $e\,P$. Thus the differential equation for this model has the form

$$\frac{dP}{dt} = r\,P\left(1 - \frac{P}{K}\right) - e\,P. \tag{5.23}$$

A constant effort would occur in industries that limit the number of licenses, or that control the number of hours a day, days per year or the number of boats that can be used to catch fish.

The right-hand-side is still a quadratic function of P and can be easily factorised. There is an unstable equilibrium at $P = 0$ and a stable equilibrium given by

$$r\left(1 - \frac{P}{K}\right) - e = 0 \tag{5.24}$$

or

$$P = (1 - \frac{e}{r})K. \tag{5.25}$$

If we assume the population to be at this stable equilibrium, then the amount of fish caught is given by

$$e\,P = e(1 - \frac{e}{r})K. \tag{5.26}$$

So what is the appropriate amount of effort e that should be used to maximise the amount of fish caught. The maximum of the RHS of Eq. (5.26) occurs when $e = \frac{1}{2}r$. That is, the optimal effort is exactly half the intrinsic growth rate of the fish. For this amount of effort the equilibrium population is $P_{eq} = K$ and the total amount of fish caught is $e\, P_{eq} = \frac{1}{4} r K$.

I hope you noticed that this is *exactly* the same as the MSY from the constant harvest model. So if the equilibrium populations of fish are the same, and the amount of fish caught is the same, what's the point of looking at both models? Is there anything different about the models?

The difference is in what happens when overfishing occurs. In the constant harvest model, if the fishermen either deliberately or accidentally exceed the MSY then there is no stable equilibrium and the fish will quickly become extinct. In the constant effort model, if the fishermen accidentally or deliberately put in too much effort, the stable equilibrium will still exist, but it will be at a smaller population. Thus by using more effort they are actually harvesting fewer fish, this sets up a natural negative feedback in the system that removes the incentive for fishermen to exceed the optimal effort.

There is another difference between the models that requires a bit of calculation. If something unforseen happens to the fish population, then it takes much longer for the population to recover and return to equilibrium in the constant harvest model compared to the constant effort model.