THE RING AXIOMS

Definition. A *ring* is a set R with an operation called *addition*:

$$\text{for any } a, b \in R, \text{ there is an element } a + b \in R,$$

and another operation called *multiplication*:

$$\text{for any } a, b \in R, \text{ there is an element } ab \in R,$$

satisfying the following axioms:
(i) Addition is associative, i.e.

$$(a + b) + c = a + (b + c) \text{ for all } a, b, c \in R.$$

(ii) There is an element of R, called the *zero element* and written 0, which has the property that

$$a + 0 = 0 + a = a \text{ for all } a \in R.$$

(iii) Every element $a \in R$ has a *negative*, an element of R written $-a$, which satisfies

$$a + (-a) = (-a) + a = 0.$$

(iv) Addition is commutative, i.e.

$$a + b = b + a \text{ for all } a, b \in R.$$

(v) Multiplication is associative, i.e.

$$(ab)c = a(bc) \text{ for all } a, b, c \in R.$$

(vi) Multiplication is distributive over addition, i.e.

$$a(b + c) = ab + ac \text{ and } (a + b)c = ac + bc \text{ for all } a, b, c \in R.$$

Definition. A *field* is a ring R which has the following extra properties:
(vii) R is commutative, i.e. $ab = ba$, $\forall a, b \in R$.
(viii) R has a nonzero identity element 1.
(ix) Every nonzero element of R is invertible.

Definition. An *integral domain* is a ring R which satisfies the following extra properties:
(vii) R is commutative.
(viii) R has a nonzero identity element 1.
(ix)' R has no zero divisors.