MATH2065
Introduction to Partial Differential Equations
Semester 2 – Tutorial Questions (Week 4)

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1. An electrical circuit in which a resistor (R) and an inductor (L) are connected in series with a voltage source ($V(t)$) has a resulting current flow $i(t)$. Using Kirchoff's Laws, it is possible to show that this current satisfies the first-order differential equation

$$L\, i'(t) + R\, i(t) = V(t).$$

   (a) It is common in circuits to have square-pulse voltage inputs, such as

$$V(t) = V_0\,[H(t-a) - H(t-b)]$$

   where V_0, a and b are positive constants ($a < b$). Given that the circuit had no initial current ($i(0) = 0$), use Laplace transforms to determine the current at a general positive time t, in terms of the parameters L, R, V_0, a and b.

   (b) Repeat part (a) if now the applied voltage is a spike $V_0\,\delta(t-a)$, where δ is the Dirac delta function.

2. Solve the ODE

$$\frac{dy}{dt} + 3y = \begin{cases} 4e^{-t}, & t < 8, \\ 2, & t > 8, \end{cases}$$

   where $y(0) = 1$, using Laplace transforms.

3. Prove that $f \star g = g \star f$, where the $\star$ denotes the convolution between functions of t.

4. Use the convolution property of Laplace transforms to determine $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$.

   <u>Hint:</u> you may find the following trigonometric identity useful:

$$\sin A \sin B = \frac{1}{2}\left[\cos(B-A) - \cos(B+A)\right].$$

5. Use the convolution property to obtain the solution to the differential equation

$$y'' - y = g(t) \quad ; \quad y(0) = 1 \;\;,\;\; y'(0) = 1\,,$$

   in terms of the function $g(t)$.

   <u>Note:</u> The answer is $y(t) = e^t + \displaystyle\int_0^t \sinh(t-\bar{t})\; g(\bar{t})\, d\bar{t}$ (a slightly different representation is also possible, if your choice for the convolution is the reverse of the order chosen to obtain this answer).

6. † Using the table of Laplace transforms, determine the Laplace transform of $\int_0^t f(\bar{t})\, d\bar{t}$ in terms of $F(s)$.

7. † In this problem, we gain some insight into the peculiar Dirac delta "function". We think of $\delta(t)$ as something which is zero everywhere except at $t = 0$, where it is "infinity". This is not a very satisfactory defintion for a function, and indeed the Dirac delta is not a function in the standard sense. A more proper definition of the Dirac delta is that it is an entity which satisfies

$$\int_a^c \delta\,(t-b)\; f(t)\,\mathrm{d}t = f(b) \quad , \quad a < b < c$$

for *any* suitably well-behaved function f. The way to think about this is that since the delta function is zero whenever its argument is not zero, it kills off most of the integral. Thus, the only contribution would occur when $t - b = 0$, i.e., when $t = b$. But this must lie in the interval of interest, (a, c), to contribute – if not, the contribution would be zero. So if $b \in (a, c)$, the Dirac delta function "picks out" the value $t = b$ in the remainder of the integrand.

   (a) For any b, investigate the value of $\int_{-1}^{3} \delta(t-b)\, t\, \cos 2t\, \mathrm{d}t$.

   (b) Show that for any function f,

$$\mathcal{L}\left\{\delta(t-b) f(t)\right\} = H(b)\, f(b)\, e^{-s\,b}$$

   where H is the unit step function.

Short answers:

1. (a) $i(t) = \dfrac{V_0}{R}\left[(1 - e^{-R(t-a)/L})H(t-a) - (1 - e^{-R(t-b)/L})H(t-b)\right]$

   (b) $i(t) = \dfrac{V_0}{L} H(t-a) e^{-R(t-a)/L}$

2. $y(t) = 2e^{-t} - 2e^{-3t} - e^{-8} H(t-8)\left(2e^{-(t-8)} - 2e^{-3(t-8)}\right) + \dfrac{2}{3} H(t-8)\left(1 - e^{-3(t-8)}\right) + e^{-3t}$

3.

4. $\dfrac{\sin t - t\,\cos t}{2}$

5. $y(t) = e^t + \int_0^t \sinh(t - \bar{t})\, g(\bar{t})\, d\bar{t}$

6. $F(s)/s$

7. (a) $b \cos 2b$ if $b \in (-1, 3)$, zero otherwise.