MATH2065
Introduction to Partial Differential Equations
Semester 2 – Tutorial Questions (Week 2)

Questions marked with the dagger symbol † are intentionally more challenging.

1. Use the method of undetermined coefficients to find a particular solution to each of the following inhomogeneous differential equations:

   (a) $y'' - 4\,y = e^x$

   (b) $y'' - 9\,y = x + 18$

   (c) $y'' - y' + 2\,y = -2\sin x$

   (d) $y'' + y = x^2$

   (e) $y'' - 9\,y = 2e^{3x}$

2. Solve the following initial value problems.

   (a) $y'' + y' - 6y = 5e^x$, with initial conditions $y(0) = 0$ and $y'(0) = 2$

   (b) $y'' + 3y' + 2y = xe^{4x}$, with initial conditions $y(0) = 0$ and $y'(0) = 1$

   (c) $y'' - 4y' + 3y = 2\sin x$, with initial conditions $y(0) = 1$ and $y'(0) = 0$

3. Show that if $y_1(x)$ is a solution of $y'' + a\,y' + b\,y = f_1(x)$ and if $y_2(x)$ is a solution of $y'' + a\,y' + b\,y = f_2(x)$, then the function $y_1(x) + y_2(x)$ is a solution of

$$y'' + a\,y' + b\,y = f_1(x) + f_2(x)\,.$$

   Use this result to obtain a particular solution of the following ODEs.

   (a) $3\,y'' + 10\,y' + 3\,y = x + \cos x$

   (b) $y'' + 5\,y' + 6\,y = \sin x + 2\,e^x$

   (c) $y'' - 4\,y' + 3\,y = e^x + e^{2x}$

4. **(Resonance)** Consider a block and spring with mass m, spring constant k, damping coefficient γ, and imposed forcing $F(t)$. Recall that its motion can be represented by

$$m\,\ddot{x} + \gamma\,\dot{x} + k\,x = F(t)\,.$$

   where $x(t)$ represents the position of the block measured from its equilibrium. For the purposes of this problem, consider the particular case where $m = 1$, $\gamma = 0$, $k = 4$ and $F(t) = \cos\omega t$, where the frequency ω is a positive constant.

   (a) Recall that the solution $x(t)$ can be represented as $x(t) = x_h(t) + x_p(t)$, where x_h and x_p are respectively the homogeneous and the particular solutions to the ODE. Determine $x_h(t)$.

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(b) Determine $x_p(t)$. [Hint: does your solution work for all positive ωs, or do you have to consider some special cases?]

(c) † Based on your solutions to (a) and (b) above, describe the behaviour of the block at large times. Is there a "special" value of the forcing frequency ω at which something different happens?

Short answers:

1. (a) $y_p = -\frac{1}{3}e^x$
   (b) $y_p(x) = -\frac{1}{9}x - 2$
   (c) $y_p(x) = -\cos x - \sin x$
   (d) $y_p(x) = x^2 - 2$
   (e) $y_p = \frac{1}{3}xe^{3x}$
2. (a) $y(x) = \frac{7}{5}e^{2x} - \frac{3}{20}e^{-3x} - \frac{5}{4}e^x$
   (b) $y(x) = -\frac{37}{36}e^{-2x} + \frac{26}{25}e^{-x} + (\frac{1}{30}x - \frac{11}{900})e^{4x}$
   (c) $y(x) = -\frac{2}{5}e^{3x} + e^x + \frac{1}{5}\sin x + \frac{2}{5}\cos x$
3. (a) $y_p = \frac{1}{3}x - \frac{10}{9} + \frac{1}{10}\sin x$
   (b) $y_p = \frac{1}{6}e^x + \frac{1}{10}\sin x - \frac{1}{10}\cos x$
   (c) $y_p = -\frac{1}{2}xe^x - e^{2x}$
4. (a) $x_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$
   (b)

$$x(t) = \begin{cases} \dfrac{1}{4-\omega^2}\cos(\omega t), & (\omega \neq 2) \\ \dfrac{t}{4}\sin(2t), & (\omega = 2) \end{cases}$$

   (c)

$$x(t) = \begin{cases} C_1 \cos(2t) + C_2 \sin(2t) + \dfrac{1}{4-\omega^2}\cos(\omega t) & (\text{if } \omega \neq 2) \\ C_1 \cos(2t) + C_2 \sin(2t) + \dfrac{t}{4}\sin(2t) & (\text{if } \omega = 2) \end{cases}$$