

# MATH2065

## Introduction to Partial Differential Equations

### Ordinary Differential Equations – Summary

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#### Homogeneous Linear ODEs

The general second-order linear homogeneous ordinary differential equation is

$$y'' + a(x)y' + b(x)y = 0.$$

- Its **general solution** takes the form

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x),$$

where  $C_1$  and  $C_2$  are constants, and  $y_1(x)$  and  $y_2(x)$  are two independent solutions.

- The general solution depends on *two* constants in general, since this is a *second-order* ODE.
- It is in general difficult to find  $y_1(x)$  and  $y_2(x)$ , if  $a$  and  $b$  are functions of the independent variable  $x$ .
- Since this is difficult, we will only deal with a simpler case in MATH2065! (See below.)

#### Homogeneous Linear Constant-Coefficient ODEs

Consider the case where the coefficients above are **constant**, i.e.,

$$y'' + a y' + b y = 0, \tag{1}$$

where  $a$  and  $b$  are real constants.

- To solve, guess a solution of the form  $y(x) = \exp(\lambda x)$ , where  $\lambda$  is not yet known. (Note:  $\exp(\lambda x)$  means  $e^{\lambda x}$ .) Substituting leads to the **characteristic equation** (also called the **auxiliary equation**)

$$\lambda^2 + a \lambda + b = 0.$$

- The characteristic equation is quadratic, and therefore there are three different situations.
  - (i) Two distinct roots: Suppose the two distinct roots are  $\lambda_1$  and  $\lambda_2$ . This means that both  $\exp(\lambda_1 x)$  and  $\exp(\lambda_2 x)$  are possible solutions. These are independent, and hence the general solution is

$$y_h(x) = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x).$$

- (ii) A repeated root: In this case, we have only *one* solution  $\exp(\lambda x)$ . It turns out that a second, independent, solution in this case is  $x \exp(\lambda x)$ . Thus, the general solution is

$$y_h(x) = C_1 \exp(\lambda x) + C_2 x \exp(\lambda x).$$

- (iii) Complex conjugate roots: Suppose these roots are  $\alpha \pm i \beta$ . Then it turns out that two independent solutions are  $\exp(\alpha x) \cos(\beta x)$  and  $\exp(\alpha x) \sin(\beta x)$ . The general solution is therefore

$$y_h(x) = C_1 \exp(\alpha x) \cos(\beta x) + C_2 \exp(\alpha x) \sin(\beta x).$$

## Inhomogeneous (Nonhomogeneous) Ordinary Differential Equations

The equation

$$y'' + a y' + b y = R(x), \quad (2)$$

is the general form for a second-order, linear, constant-coefficient, inhomogeneous (nonhomogeneous) ODE. The function  $R(x)$  is the *inhomogeneity*. To solve this:

- (i) First, solve the *homogeneous* version of the equation (2) by setting  $R(x) \equiv 0$ ; the technique for equation (1) will yield the **homogeneous solution** in the form

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x).$$

- (ii) Determine a **particular solution**  $y_p(x)$  to (2) (by using, say, undetermined coefficients as described below).

- (iii) The general solution to (2) is then

$$y(x) = y_h(x) + y_p(x).$$

### Method of Undetermined Coefficients

Used for finding particular solutions  $y_p(x)$  to equations of the form (2). *Guess* the following type of form for  $y_p(x)$ , substitute into (2), and then equate coefficients to find the **undetermined coefficients**.

$R(x)$	$y_p(x)$
$k \exp(\alpha x)$	$C \exp(\alpha x)$
$P_n(x)$ polynomial degree $n$	$a_n x^n + \cdots + a_1 x + a_0$
$k \cos(\omega x)$ or $k \sin(\omega x)$	$A \cos(\omega x) + B \sin(\omega x)$
$k \exp(\alpha x) \cos(\omega x)$ or $k \exp(\alpha x) \sin(\omega x)$	$A \exp(\alpha x) \cos(\omega x) + B \exp(\alpha x) \sin(\omega x)$
$k P_n(x) \exp(\alpha x)$	$\exp(\alpha x) [a_n x^n + \cdots + a_1 x + a_0]$

Modification Rule: If even one term in the guessed form is part of the homogeneous solution, multiply *everything* by  $x$ . If the same is still true, multiply everything by  $x$  again.