MATH 2065 — LECT 3.1

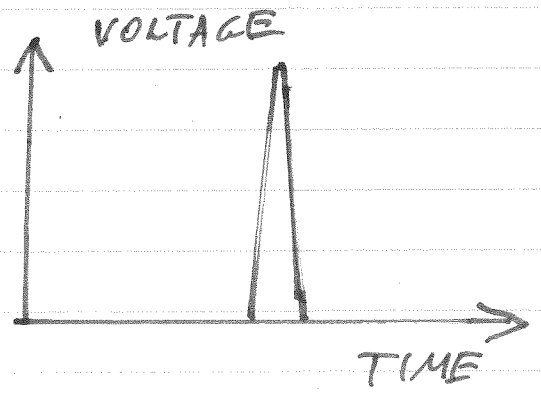
DIRAC'S DELTA FUNCTION — SHORT IMPULSES

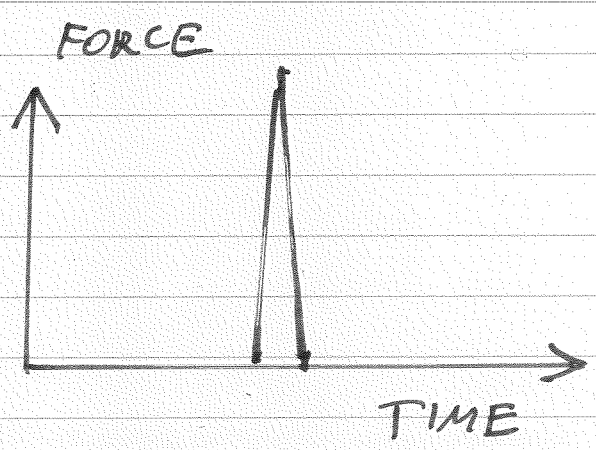
Phenomena of an IMPULSIVE nature such as the action of strong forces or voltages over SHORT intervals of time arise in various applications:

Examples

A mechanical system is hit by a hammer blow, a tennis ball is hit by a racket or a bat hits a baseball, and so on.

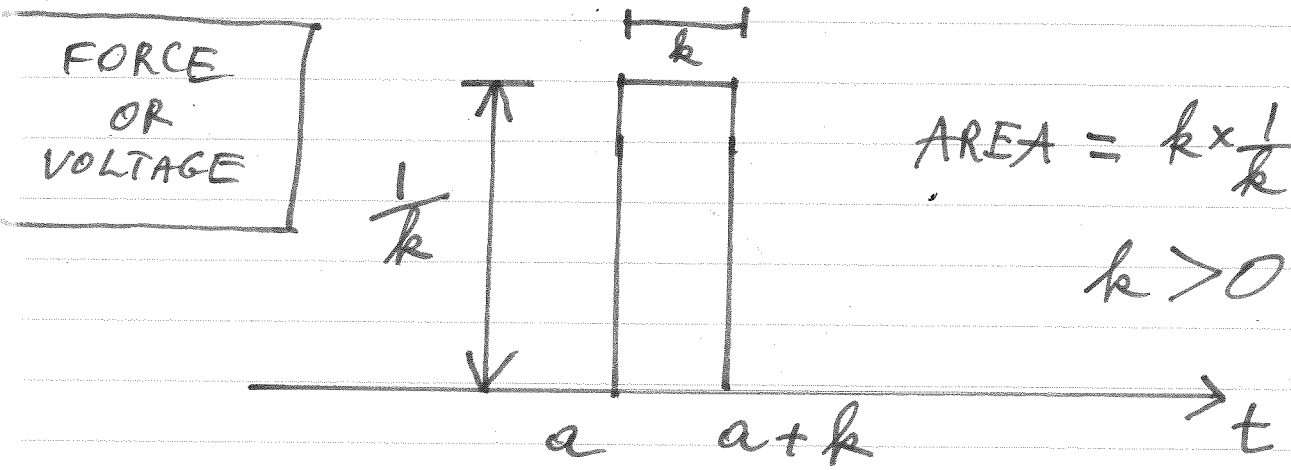
Such problems are MODELED by the Dirac Delta function and can be solved very efficiently by the Laplace transform





To model situations of this type, we consider the function

$$\boxed{f_k(t-a) = \begin{cases} 1/k & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases} \quad (1)}$$



$$\text{AREA} = k \times \frac{1}{k} = 1$$

$$k > 0$$

We now take the limit of f_k as $k \to 0$ and call this limit $\delta(t-a)$, that is:

$$\boxed{\delta(t-a) = \lim_{k \to 0} f_k(t-a)}$$

$\delta(t-a)$ is called the <u>DIRAC DELTA FUNCTION</u> or the <u>UNIT IMPULSE FUNCTION</u>

$\delta(t-a)$ is not a function in the ordinary sense as used in calculus, but is a so-called GENERALIZED FUNCTION.

In physics, the integral of a force acting over a short time interval is called the

IMPULSE OF THE FORCE. I_k

i.e:

$$\boxed{I_k = \int_0^{\infty} f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1 \quad (2)}$$

Taking the limit of (1) and (2) as $k \to 0$

we get:

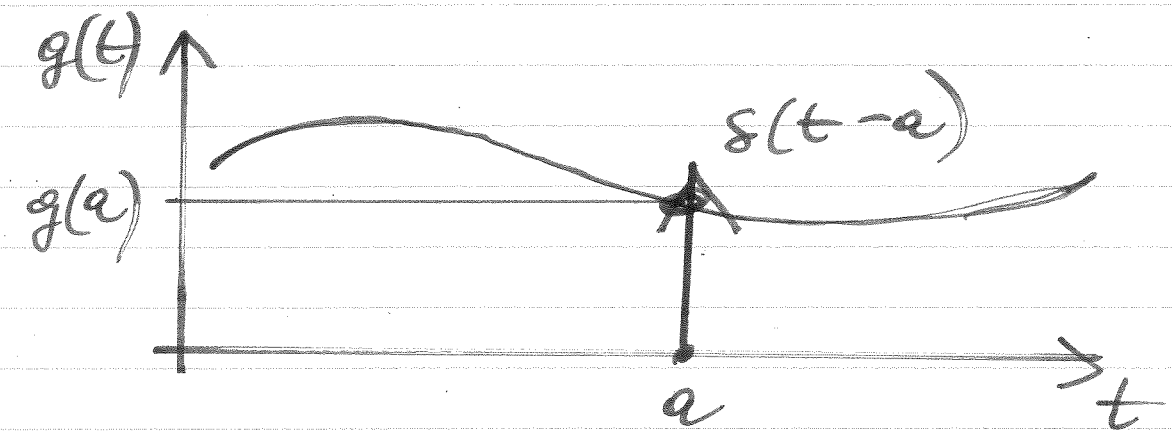
$$\boxed{\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}} \quad \text{AND} \quad \boxed{\int_0^{\infty} \delta(t-a)\,dt = 1}$$

$\delta(t)$ has INFINITE HEIGHT, ZERO WIDTH, And AREA $= 1$

SIFTING PROPERTY OF $\delta(t-a)$

If $g(t)$ is a continuous function then:

$$\boxed{\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)}$$



That is, $\delta(t-a)$ "picks out" the value of $g(t)$ at $t=a$ when we integrate $g(t)\,\delta(t-a)$.

Laplace Transform of $\delta(t-a)$

Using the definition of Laplace transform

$$\mathcal{L}[\delta(t-a)] = \int_0^\infty \delta(t-a)\, e^{-st}\, dt$$

$$= e^{-as} \qquad \left\{ \begin{array}{c} a>0 \\ s>0 \end{array} \right\}$$

Example Hammer blow response of a mass – spring system.

Solve:

$$y'' + 3y' + 2y = \delta(t-1)$$

$$y(0) = 0 \; ; \; y'(0) = 0$$

Take L.T to both sides of the ODE

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = e^{-s} \quad (a=1)$$

$$[s^2 Y(s) - 0 - 0] + 3[s\,Y(s) - 0] + 2\,Y(s) = e^{-s}$$

$$Y(s)[s^2 + 3s + 2] = e^{-s}$$

$$Y(s) = \frac{e^{-s}}{s^2+3s+2}$$

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+3s+2}\right]$$

Use Partial Fraction expansion on:

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

$$\Rightarrow \quad A(s+2) + B(s+1) = 1$$

Let $s = -2 \Rightarrow B(-1) = 1 \Rightarrow$

$$\boxed{B = -1}$$

Let $s = -1 \Rightarrow A(1) = 1$

$$\boxed{A = 1}$$

$$\Rightarrow \boxed{\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}.}$$

3.1.7

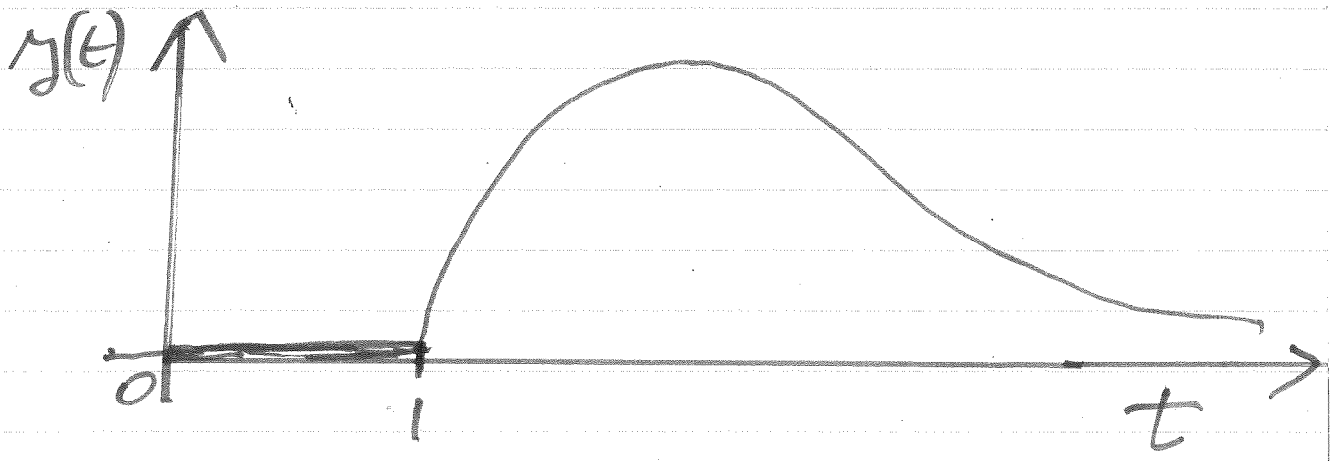
$$y(t) = \mathcal{L}^{-1}\left[\frac{e^{-s}}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}}{s+2}\right]$$

Use t – shifting from the table

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+1}\right] = H(t-1)e^{-(t-1)}$$

$$\mathcal{L}\left[\frac{1}{s+2}\right] = e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+2}\right] = H(t-1)e^{-2(t-1)}$$

$$\therefore \boxed{y(t) = H(t-1)\left[e^{-(t-1)} - e^{-2(t-1)}\right]}$$