MATH 2065 - LECT 1.3

NONHOMOGENEOUS ODEs

In this section we learn how to solve second-order NONHOMOGENEOUS linear ODEs with constant coefficients:

$$\boxed{y'' + a y' + b y = G(x)}$$

a, b are constants and $G(x)$ is a continuous function.

Examples

1) $y'' + y' - 2y = x^2$

2) $y'' + 4y = e^{3x}$

3) $y'' + y' - 2y = \sin x$

In this course we use the method of UNDETERMINED COEFFICIENTS to solve the ODE.

To find the <u>GENERAL SOLUTION</u> of

$$\boxed{y'' + ay' + by = G(x)} \quad (1)$$

there are 3 STEPS:

1) Find the general solution of the homogeneous ODE: $\boxed{y'' + ay' + by = 0}$

i.e: with RHS $= 0$

Call this solution $y_h(x)$

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2) Find a PARTICULAR SOLUTION of $\boxed{y'' + ay' + by = G(x)}$

using the method of <u>UNDETERMINED COEFFICIENTS</u>. Call this $y_p(x)$

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3) The general solution is then the sum of $y_h(x)$ and $y_p(x)$

i.e: $$\boxed{y(x) = y_h(x) + y_p(x)}$$

EXAMPLE 1

a) Find a PARTICULAR SOLUTION (1)

of the nonhomogeneous ODE $\boxed{y'' + y' - 2y = x^2}$

b) Find the GENERAL SOLUTION of the corresponding HOMOGENEOUS ODE (2)

$$\boxed{y'' + y' - 2y = 0}$$

c) Find the GENERAL SOLUTION of the original nonhomogeneous ODE.

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a) Using the method of UNDETERMINED COEFFICIENTS, since the RHS is a second order polynomial, we TRY as our guess the second order polynomial

$$\boxed{y_p(x) = Ax^2 + Bx + C}$$

To find A, B, C we substitute into the ODE (1):

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$\underbrace{2A}_{y''} + \underbrace{2Ax+B}_{y'} - \underbrace{2(Ax^2+Bx+C)}_{-2y} = x^2$$

Expanding and simplifying :

$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$

i.e.:

$$-2Ax^2 + (2A-2B)x + (2A+B-2C) = x^2$$

Equating the coefficients of like powers of x, gives

$$-2A = 1 \qquad 2A - 2B = 0$$

$$2A + B - 2C = 0$$

Solving simultaneously gives

$$A = -1/2 \; ; \quad B = -1/2 \; ; \quad C = -3/2$$

Therefore, a particular solution is:

$$\boxed{y_p(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - 3/4}$$

b) Solving the homogeneous ODE (2):

$$\lambda^2 + \lambda - 2 = 0 \quad \Rightarrow \quad (\lambda - 1)(\lambda + 2) = 0$$

$$\lambda = 1, -2$$

$$\therefore \quad \boxed{y_h(x) = Ae^{x} + Be^{-2x}}$$

c) The general solution of (1) is:

$$y(x) = y_h(x) + y_p(x)$$

i.e.:

$$\boxed{y(x) = Ae^{x} + Be^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - 3/4}$$

A, B arbitrary constants.

If we are given extra conditions like

$$y(0) = 1 \qquad y'(0) = 0$$

then we can find values for A, B.

$$\left.\begin{aligned} y(0) &= A + B - 3/4 = 1 \\ y'(0) &= A - 2B - 1/2 = 0 \end{aligned}\right\} \text{Solving simultaneously gives}$$

$$\left.\begin{aligned} A + B &= 7/4 \\ A - 2B &= 1/2 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 4/3 \\ B &= 5/12 \end{aligned}$$

Therefore, the solution that satisfies the two initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0$$

is :

$$\boxed{y(x) = \frac{4}{3} e^{x} + \frac{5}{12} e^{-2x} - \frac{1}{2} x^{2} - \frac{1}{2} x - 3/4}$$

EXAMPLE 5 Repeat the steps of Example 1 using the ODE:

$$\boxed{y'' + 4y = e^{3x}} \quad (1)$$

1) Particular solution: Here we try

$$y_p = Ae^{3x} \quad \text{substitute into ODE}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$\longrightarrow \quad 9Ae^{3x} + 4Ae^{3x} = e^{3x}$$

$$\longrightarrow \quad 13Ae^{3x} = e^{3x} \longrightarrow \boxed{A = 1/13}$$

$$\therefore \quad \boxed{y_p(x) = \frac{1}{13} e^{3x}}$$

2) General solution of $y'' + 4y = 0$

$$\lambda^2 + 4 = 0 \longrightarrow \lambda = \pm 2i$$

$$\therefore \quad \boxed{y_h(x) = A\cos 2x + B\sin 2x}$$

3) General solution of (1) is:

$$y(x) = y_h + y_p = A\cos 2x + B\sin 2x + \frac{1}{13} e^{3x}$$