COMPLEX FOURIER SERIES

Here we express a Fourier series in terms of complex functions. Very often the algebra is simplified and it is also a way to introduce FOURIER TRANSFORMS later on.

Recall the general FS for a periodic function $f(x+2L) = f(x)$ of period $2L$:

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$$

with the Fourier coefficients: ⊛

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$ ⊛⊛

Also recall Euler's formula

$$\boxed{\begin{aligned} \cos\theta + i\sin\theta &= e^{i\theta} \quad (1) \\ \cos\theta - i\sin\theta &= e^{-i\theta} \quad (2) \end{aligned}}$$

Adding (1) + (2) gives

$$\boxed{\cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]} \quad (3)$$

Subtracting (1) − (2) gives

$$\boxed{\sin\theta = \frac{1}{2i}\left[e^{i\theta} - e^{-i\theta}\right]} \quad (4)$$

Substituting (3) & (4) into the FS ⊗

with $\boxed{\theta = \frac{n\pi x}{L}}$

gives:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}\right) + b_n \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}\right)\right]$$

REARRANGING:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)\, e^{\frac{in\pi x}{L}} + \frac{1}{2}(a_n + ib_n)\, e^{\frac{-in\pi x}{L}}\right] \quad \circledast\circledast\circledast$$

using $\frac{1}{i} = -i$

Introduce the notation

$$c_0 = a_0 \quad (5) \qquad c_n = \frac{1}{2}(a_n + ib_n) \quad (6)$$

$$d_n = \frac{1}{2}(a_n - ib_n) \quad (7)$$

$$n = 1, 2, 3 \ldots$$

and substitute the expressions $\circledast\circledast$ for the Fourier coefficients into (5), (6), (7) to obtain:

$$c_0 = a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$c_n = \frac{1}{2}(a_n + i b_n)$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{in\pi x}{L}}\,dx$$

$$d_n = \frac{1}{2}(a_n - i b_n)$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi x}{L}}\,dx$$

Comparing c_n and d_n we notice that

$$\boxed{d_n = c_{-n}}$$

Therefore ⊛⊛⊛ becomes:

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_{-n} e^{\frac{in\pi x}{L}} + c_n e^{-\frac{in\pi x}{L}} \right)$$

$$= c_0 + \sum_{n=-\infty}^{-1} c_n e^{-\frac{in\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty} c_n e^{-\frac{in\pi x}{L}}$$

Now, we can combine all three terms into one sum:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}}}$$

where

$$\boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{\frac{in\pi x}{L}} \, dx}$$

This is the COMPLEX FORM OF the FOURIER SERIES ⊛ & ⊛⊛.

OBTAINING THE REAL FS

Since we started with a real function $f(x)$ we may derive the real FS from the complex FS.

Recall that:

$$c_n = \frac{1}{2}(a_n + i b_n) \quad (8)$$

$$c_{-n} = \frac{1}{2}(a_n - i b_n) \quad (9)$$

Therefore:

adding (8) + (9) gives

Subtracting (8) − (9) gives

$$\boxed{\begin{aligned} a_n &= c_n + c_{-n} \\ b_n &= \frac{1}{i}(c_n - c_{-n}) \\ a_0 &= c_0 \end{aligned} \quad (10)}$$

So we can now construct the real FS:

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]}$$

where the Fourier coefficients are given by expressions (10)

EXAMPLE

Find the complex FS of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$$f(x+2\pi) = f(x) \qquad (L = \pi)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (-1)\, e^{inx}\, dx + \frac{1}{2\pi}\int_{0}^{\pi} (1)\, e^{inx}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-e^{inx}}{in}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{e^{inx}}{in}\right]_{0}^{\pi}$$

$$= \frac{1}{2in\pi}\left[-1 + e^{-in\pi} + e^{in\pi} - 1\right]$$

$$= \frac{1}{2in\pi}\left[-2 + 2\underbrace{\left(\frac{e^{in\pi} + e^{-in\pi}}{2}\right)}_{\cos(n\pi) = (-1)^n}\right]$$

$\therefore$

$$c_n = \frac{2}{2in\pi}\left[\cos(n\pi) - 1\right]$$

$$= \begin{cases} 0, & n \text{ EVEN} \\ \dfrac{-2}{in\pi}, & n \text{ ODD} \\ & (= \ldots -3, -1, 1, 3 \ldots) \end{cases}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{2\pi}\int_{0}^{\pi}(1)\,dx$$

$$= \frac{1}{2\pi}\Big[-x\Big]_{-\pi}^{0} + \frac{1}{2\pi}\Big[x\Big]_{0}^{\pi}$$

$$= \frac{-1}{2\pi}(\pi) + \frac{1}{2\pi}(\pi) = 0$$

Therefore, the complex FS becomes

$$\boxed{f(x) = \frac{2i}{\pi}\sum_{n=(\ldots -3,-1,1,3\ldots)} \frac{1}{n} e^{-inx}}$$

since $\frac{-1}{i} = +i$

We may now calculate the real FS using expressions (10)

$$a_0 = c_0 = 0$$

$$a_n = c_n + c_{-n} = \frac{2i}{n\pi} - \frac{2i}{n\pi} = 0$$

$$b_n = \frac{1}{i}(c_n - c_{-n}) = \frac{1}{i}\left(\frac{2i}{n\pi} + \frac{2i}{n\pi}\right)$$

$$= \frac{4}{n\pi} \qquad n = 1, 3, 5 \ldots$$

Direct calculation gives

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} (1)\sin(nx)\,dx + \int_{-\pi}^{0} (-1)\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{\cos(n\pi)}{n}\right]_{-\pi}^{0}$$

$$= \frac{1}{n\pi}\left[1 - (-1)^n\right] + \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0, & n \text{ EVEN} \\ \dfrac{4}{n\pi}, & n \text{ ODD} \end{cases}$$

which is the same as above.