Fourth Year Pure Mathematics

2011 Handbook

School of Mathematics and Statistics
University of Sydney

THE UNIVERSITY OF SYDNEY

www.maths.usyd.edu.au/u/UG/HM/

CONTENTS

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something completely different.
Goethe, *Maximen und Reflexionen*

CHAPTER 1

The structure of pure mathematics four

1.1. Introduction

In linguistics it is increasingly believed that universal features of language are reflections of the structure of the human brain and its perception of the world around us. In a similar fashion, mathematics is a universal language that has been developed to understand and describe how nature and life work. Mathematics, both in structure and development, is inextricably bound to our attempts to understand the world around us and our perceptions of that world. We see this in the mathematical descriptions and formulations of models in the theoretical and applied sciences: from physics, computer science and information theory on the one hand, to engineering, chemistry, operations research and economics on the other.

Just as remarkable is the way in which esoteric and abstract mathematics finds applications in the applied sciences. Indeed, one of the most exciting developments in science over the past decade has been the re-emergence of a dynamic interaction between pure mathematicians and applied scientists, which is bringing together several decades of the relatively abstract and separate development of pure mathematics and the sciences. Examples include the applications of singularity theory and group theory to symmetry-breaking and bifurcation in engineering; number theory to cryptography; category theory and combinatorics to theoretical and computational computer science; and, most spectacular of all, the recent developments of general field theories in mathematical physics based on the most profound work in complex analysis and algebraic geometry. Of course, this interaction is not one way. For example, there is the recent discovery of "exotic" differential structures on $\mathbb{R}^4$ utilising ideas from Yang-Mills theory.

There are many valid approaches to the study of Pure Mathematics in the final Honours Year. Thus, the course may be regarded as useful in its own right, or may lead on to an M. Sc or Ph.D. or to a teaching position in University or High School. In another direction, what want a solid base from which to continue with studies in computer science or physics, for example. Finally, you may intend to seek employment with the CSIRO or in the operations research field, or in a financial institution. In the latter circumstances, one well-known advantage of studying mathematics is that mathematics gives training in a particular way of thinking and an analytical approach to problem solving. Mathematicians are highly adaptable (and employable).

The Fourth Year Honours program in Pure Mathematics caters for the various needs described above by offering a highly flexible and adaptable program, which is both interesting and challenging. We offer a combination of core courses, which introduce the major areas of mathematics, together with a smorgasbord of deeper courses courses which can be arranged to suit your personal requirements.

In brief, the Fourth Year course comprises the equivalent of six lecture courses, together with an essay project (the equivalent of four lecture courses) and a 30 minute talk on the essay project.

A description of the various components of the course is given below. For detailed descriptions of the courses, the essay project, etc., see the appropriate chapter in this Handbook.

1.2. The lecture courses

Students are required to be assessed on 6 units of approved lecture courses (or equivalent - see below).

In 2011, the courses may be chosen from:

a) three PM4 core courses, each worth 1 unit;
b) other PM4 courses, each worth 1 unit.
   (These may presume some knowledge of one or more of the core courses.)
c) third year *advanced* courses, each worth 1 unit.
   (Students in Pure Mathematics 4 may take any 3(A) course which they have not previously taken.)
d) Approved substitutions (up to the value of 2 units) by courses given by other Departments. (See §1.5 below.)
e) Reading courses arranged with staff members (after consultation with the PM4 coordinator).

Read carefully the guidelines in §1.5 below.

In addition, we are negotiating with UNSW over the possibility of sharing some PM4 Courses. (The likely UNSW courses include “Graph Theory" and “Banach Algebras"; the potential difficulties are in timetabling.)

Overall, the lecture courses offered at the level of PM4 and above are intended to introduce students to the major divisions of modern mathematics and provide a knowledge of some of the main ideas needed for the understanding of much of contemporary mathematics, while still reflecting the research interests within the pure mathematics research groups.

The “core" of Fourth Year is considered to include Commutative Algebra, Functional Analysis and Algebraic Topology. Students are strongly advised to take all of the core courses.

1.3. Pure Mathematics 4/PG Courses for 2011

All lectures are in Carslaw Room 830.

Semester I	
Algebraic Topology - Mon, Wed, 9am	Hillman
Functional Analysis - Mon, Wed, 10am	Parkinson
Lie algebras - Tue, Thu, 9am	A. Henderson
Commutative Algebra - Tue, Thu, 11am	Lehrer
Semester II	
Modular representation theory	Mathas
Spectral theory and PDEs	Dancer
Banach-Tarski and Amenability	Thomas
TBA	Gamburd

If you are unsure about the combination of courses you should take, consult with your supervisor or the course coordinator. In any case, you are very welcome to attend all the lecture courses.

1.4. Pure Mathematics 3(A) Courses for 2011

SEMESTER I	SEMESTER II
Algebra and Number Theory (3962)	Modules and Group Representations (3966)
Metric Spaces (3961)	Measure Theory and Fourier Analysis (3969)
	Differential Geometry (3968)

1.5. Tailor-made courses

Students may also take certain courses of an essentially mathematical nature other than these (for instance, in Applied Mathematics, Econometrics, Formal Logic, Statistics, etc.) **with the approval** of the Course Coordinator. (Normally we require that at least four of the six course units be chosen from the Pure Mathematics options). Details of the Applied Mathematics 4 and Mathematical Statistics 4 options may be obtained from the Coordinators, Dr Martin Wechselberger (Applied Maths - Carslaw 628, tel. 9351 3860, *M.Wechselberger@maths.usyd.edu.au*) and Dr Michael Stewart

(Math. Stats. Carslaw 818, tel. 9351 5765, *M.Stewart @maths.usyd.edu.au*).

A number of staff are usually willing to supervise a reading course in their particular area of interest. Consult the course coordinator if you have a special topic in mind that might be acceptable as a reading course. Reading courses are generally a matter between the student and a willing member of the department, subject to the approval of the course coordinator.

If you wish to do a **reading course** in Pure Mathematics, or substitute a course from outside Pure Mathematics, you should ask the lecturer to prepare a **summary** of the course and description of the **assessment**. This should then be submitted to the course coordinator for approval. Provided the guidelines sketched above are followed, and a satisfactory balance overall in the Fourth year courses is maintained, approval will normally be granted.

1.6. The essay project

The essay project counts as the equivalent of four PM4 units. Work on the essay project proceeds throughout the year and the finished essay is submitted near the end of the second semester. Note that it is also possible for the project to be supervised by a member of another department (or jointly).

1.7. The talk

As part of the essay project, students are required to give a talk about their project. Talks are normally scheduled to take place in September.

1.8. Mathematics in other languages

Ability to read mathematics in at least one approved foreign language is no longer a requirement for Pure Mathematics 4. An increasing proportion of mathematical papers are now written in English, there are still a significant number of important mathematical works not written in English. In addition, many older mathematical works written in other languages have not yet been translated into English. For these reasons, students who are seriously thinking of pursuing further studies in mathematics are strongly encouraged to acquire a reading knowledge of mathematics in at least one foreign language. Indeed, such knowledge is often a requirement of M. Sc. and Ph.D. courses in mathematics (especially in the U. S. A.).

In particular, a working knowledge of mathematical French is extremely useful (and relatively easy to acquire). At present, Russian is less useful, as the collapse of the Soviet system has obliged many of the best mathematicians from the former Soviet Union to work in the West, and in any case there are English translations of most of the Russian language mathematics journals. (However, our library still subscribes to the Russian originals in most cases, as they are much cheaper than the translations). On the other hand German may again become the preferred language of publication for mathematicians in an united Germany. Although the mathematical schools of China and Japan are large and increasingly important acquiring a reading knowledge of the languages of these countries is difficult and time-consuming for most adults.

The Departments of French and German offer reading courses that enable one to acquire a working reading knowledge of French and German. These courses begin in the first semester and participants must register with the relevant Department before the start of the course.

I fear that I bore you with these details, but I have to let you see my little difficulties, if you are to understand the situation.

Sherlock Holmes, *A Scandal in Bohemia*

CHAPTER 2

Entry, administration and assessment

2.1. Entry Requirements for Pure Mathematics 4

Students who have fulfilled the requirements of the faculty in which they are enrolled and satisfied conditions (a), and (b) or (c) below are eligible to enrol in Pure Mathematics 4:

a) taken 24 credit points of third year mathematics units (see the senior pure and applied mathematics handbook) with at least 16 of these credit points in pure mathematics;

b) obtained a distinction average or better in 24 credit points of third year mathematics units

or,

c) obtained a credit average in 24 credit points of third year mathematics units, *including* a credit in at least one Pure Mathematics 3 *Advanced* unit.

Entry to PM4 is also subject to the approval of the Head of School

Note Since we advise all PM4 students to take the core courses (commutative algebra, algebraic topology and functional analysis), the natural prerequisites for PM4 are: Metric Spaces; Algebra 1; and Lebesgue and Fourier Analysis. Students without this background should expect to do some preliminary reading over the summer (see the course coordinator for advice if necessary).

2.2. Actions to be taken

All students intending to take Pure Mathematics 4 in 2011 should see the *PM4 Course Coordinator*, Dr Laurentiu Paunescu (Carslaw 721, tel. 9351-2969, e-mail: *laurent@maths.usyd.edu.au*) at their earliest opportunity, and in any case well before the beginning of the new teaching year. The Course

Coordinator will advise you about choosing a supervisor and a topic for the essay project (see also section 4.2).

2.3. Administrative arrangements

The Course Coordinator is in charge of Pure Mathematics 4 and should be consulted about any organisational problems that may arise.

In particular, students should note that the Course Coordinator's permission should be obtained if you wish to substitute courses from outside, or take a reading course or a postgraduate course. In the first instance, however, you should discuss such matters with your supervisor. Provided you can agree, the Course Coordinator's permission would *normally* be a formality.

Please take particular note of the procedure to be followed if you are sick or other circumstances arise that may lead to late submission of your essay (see §4.4). Also note that at the end of first semester a progress report must be given to the Course Coordinator (see Chapter 5).

When we know that you are enrolled for PM4 you will be given a computer account. *The usual way in which messages for PM4 students will be distributed will be via e-mail*. Please remember to check your e-mail regularly. This will become second nature once you start to type up your essays.

2.4. Assessment

The possible results for Fourth year are First Class Honours, Second Class Honours division 1, Second Class Honours Division 2, Third Class Honours and No Award (Fail), usually abbreviated I, II-1, II-2, III and F. The last two are rarely awarded.

Each Fourth Year course is assessed at a time and in a manner arranged between the lecturer and the class. Usually, a written examination is held during the exam period immediately following the course; however, some courses are assessed entirely by assignment. It is undesirable to have examinations during term or to have many papers deferred to the end of the year. Each PM3 advanced course is assessed in the usual way. Students are informed about their performance as information becomes available.

Marks provided by the lecturers are scaled according to the lecturer's judgement as to what constitutes a First Class, II-1, etc., performance on each course. The marks from the best 6 units are counted towards the final mark.

The essay is equivalent to 4 PM4 units; it accounts for 40% of the years assessment.

As well as assessing the Fourth Year performance, the Department is required to make a recommendation for a grade of Honours based on the performance in all subjects over the four years. In exceptional cases, the grade of Honours awarded could differ from the level of performance in the Fourth Year.

2.5. Honours grades

The Faculty of Science has given the following guidelines for assessment of student performance in fourth year.

95–100 Outstanding First Class quality of clear Medal standard, demonstrating independent thought throughout, a flair for the subject, comprehensive knowledge of the subject area and a level of achievement similar to that expected by first rate academic journals. This mark reflects an exceptional achievement with a high degree of initiative and self-reliance, considerable student input into the direction of the study, and critical evaluation of the established work in the area.

90-94 Very high standard of work similar to above but overall performance is borderline for award of a Medal. Lower level of performance in certain categories or areas of study above.

Note: An honours mark of 90+ and a third year WAM of 80+ are necessary but not sufficient conditions, for the award of the Medal. Examiners are referred to the Academic Board Guidelines on the award of Medals found in the general policy pages at the front of the Examiners' Manual.

80-89 Clear First Class quality, showing a command of the field both broad and deep, with the presentation of some novel insights. Student will have shown a solid foundation of conceptual thought and a breadth of factual knowledge of the discipline, clear familiarity with and ability to use central methodology and experimental practices of the discipline, and clear evidence of some independence of thought in the subject area. Some student input into the direction of the study or development of techniques, and critical discussion of the outcomes.

75-79 Second class honours, first division - student will have shown a command of the theory and practice of the discipline. They will have demonstrated their ability to conduct work at an independent level and complete tasks in a timely manner, and have an adequate understanding of the background factual basis of the subject. Student shows some initiative but is more reliant on other people for ideas and techniques and project is dependent on supervisor's suggestions. Student is dedicated to work and capable of undertaking a higher degree.

70-74 Second class honours, second division - student is proficient in the theory and practice of their discipline but has not developed complete independence of thought, practical mastery or clarity of presentation. Student shows adequate but limited understanding of the topic and has largely followed the direction of the supervisor.

65-69 Third class honours - performance indicates that the student has successfully completed the work, but at a standard barely meeting honours criteria. The student's understanding of the topic is extremely limited and they have shown little or no independence of thought or performance.

The award of a medal is *not* made just on the basis of a numerical mark or formula. The merits of each eligible candidate are debated by the Board of Examiners of the relevant Faculty.

2.6. School Facilities

Pure Mathematics 4 students traditionally enjoy a number of privileges. These include:

- Office space and a desk in the Carslaw Building.
- A computer account with access to e-mail and the World-Wide Web, as well as $\TeX$ and laser printing facilities for the preparation of essays and projects.
- A photocopying account paid by the School for essay/project source material.
- After-hours access to the Carslaw Building. (A deposit is payable.)
- A pigeon-hole in room 728 - please inspect it regularly as lecturers often use it to hand out relevant material.
- Participation in the School's social events.

- Class representative at School meetings.

2.7. Scholarships, Prizes and Awards

The following scholarships and prizes may be awarded to Pure Mathematics 4 students of sufficient merit. (Note that unless the conditions of the prize state otherwise, as in the David G.A.Jackson Prize and the A.F.U.W. Prize, these prizes are also open to all Honours students in the School of Mathematics and Statistics.)

2.7.1. Joye Prize in Mathematics. Value: $5000

To the most outstanding student completing fourth year honours in the School of Mathematics and Statistics, $5000 plus medal and shield.

2.7.2. George Allen Scholarship in Pure Mathematics. Value:$400

To a student proceeding to Honours in Pure Mathematics who has shown greatest proficiency in at least 24 credit points of Senior units of study in the School of Mathematics and Statistics.

2.7.3. Barker Prize. Value: $375

Awarded at the fourth (Honours) year examiner's meetings for proficiency in Pure Mathematics, Applied Mathematics or Mathematical Statistics.

2.7.4. Ashby Prize. Value: $360

Offered annually for the best essay, submitted by a student in the Faculty of Science, that forms part of the requirements of Pure Mathematics 4, Applied Mathematics 4 or Mathematical Statistics 4.

2.7.5. Norbert Quirk Prize No IV. Value: $250

Awarded annually for the best essay on a given mathematical subject by a student enrolled in a fourth year course in mathematics (Pure Mathematics, Applied Mathematics or Mathematical Statistics) provided that the essay is of sufficient merit.

2.7.6. David G.A.Jackson Prize. Value: $1000

Awarded for creativity and originality in any undergraduate Pure Mathematics unit of study.

2.7.7. Australian Federation of Graduate Women Prize in Mathematics. Value: $175

Awarded annually, on the recommendation of the Head of the School of Mathematics and Statistics, to the most distinguished woman candidate for the degree of BA or BSc who graduates with first class Honours in Pure Mathematics, Applied Mathematics or Mathematical Statistics.

2.7.8. University Medal.

Awarded to Honours students who perform outstandingly. The award is subject to Faculty rules, which require a Faculty mark of 90 or more in Pure Mathematics 4 and a Third year WAM of 80 or higher. More than one medal may be awarded in any year.

I dislike arguments of any kind. They are always vulgar,
and often convincing.
Oscar Wilde, *The importance of being earnest*

CHAPTER 3

Course descriptions

All courses offered in 2011 are 2 lecture per week courses and count as 1 unit. Some of the Semester II courses have one of the core courses as a prerequisite. In addition, all 3(A) courses, not previously examined, are be available for credit. Each 3(A) course is run at 3 lectures per week (plus tutorial(s)) and counts as 1 unit. For substitutions by courses not given by Pure Mathematics see Section 1.3.

3.1. Fourth Year Courses — Semester I

Algebraic Topology — J. Hillman

Algebraic topology advanced more rapidly than any other branch of mathematics during the twentieth century. Its influence on other branches, such as algebra, algebraic geometry, analysis, differential geometry and number theory has been enormous.

The typical problems of topology such as whether $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ or whether the projective plane can be embedded in $\mathbb{R}^3$ or whether we can choose a continuous branch of the complex logarithm on the whole of $\mathbb{C}\backslash\{0\}$ may all be interpreted as asking whether there is a suitable continuous map. The goal of Algebraic Topology is to construct invariants by means of which such problems may be translated into algebraic terms. The homotopy groups $\pi_n(X)$ and homology groups $H_n(X)$ of a space X are two important families of such invariants. The homotopy groups are easy to define but in general are hard to compute; the converse holds for the homology groups.

In past years the PM4 course *Algebraic Topology* has run at 3 lectures per week, and we have been able to consider both homology and fundamental group. This year (2011) the course shall run at 2 lectures per week, and we shall consider only the homological approach. (Aspects of the fundamental group may be considered by Dr Thomas in her Semester II course on amenability.) We begin with simplicial homology theory. Then we define singular homology theory, and over several weeks develop the properties which are summarized in the Eilenberg-Steenrod axioms. (These give an axiomatic characterization of homology for reasonable spaces.) We then apply homology to various examples, and may also touch upon differential forms and de Rham cohomology in the final week. Although we shall assume no prior knowledge of Category Theory, we shall introduce and use categorical terminology where appropriate. (Indeed Category Theory was largely founded by algebraic topologists.)

Assessment: There shall be 2 assignments, together worth 20% of the assessment, and a final exam, worth 80%. The assignments and exam shall be based upon the exercises in the course notes.

Text? There is no set text for this course. However I have put typed notes (corresponding closely to the content of the course) on the web: go to the School "Teaching" page and follow the links through “Honours" and “Algebraic Topology". Allan Hatchers’s book *Algebraic Topology* (Cambridge University Press, 2002) is very good, although it contains much more than we can cover in the time available. (It is also available on the web, through “www.math.cornell.edu/$\sim$hatcher".)

Prerequisite: *Metric Spaces*.

Commutative Algebra — G. Lehrer

General: One of the most significant mathematical innovations of the 20th century was the development of “context-free geometry”. The key idea of this is that the study of the simultaneous solutions of polynomial equations such as $x^4 + y^4 + z^4 - 5x^2y^2 = 0$ may be carried out in a way which is independent of the domain in which the variables $x, y, z, \ldots$ lie. For example they may lie in $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ or $\mathbb{F}_q$, and prior to these developments the study of solutions in those domains would have been regarded as separate disciplines. Thus new common ground now exists between geometry over various domains. There is also now a much better understanding of much studied concepts such as the “multiplicity” of a higher order intersection of curves or their higher dimensional analogues.

The foundations for these spectacular advances were laid largely by the Paris school of Grothendieck in the 1950’s and 60’s building on the work of many mathematicians over many centuries, but most importantly on that of Hilbert in the 1920’s. Its basis is the abstraction of geometry by algebra, and this course, on commutative algebra, is intended to be an introduction to the basic ideas of the subject. The course will cover the basic concepts of commutative algebra, and illustrate them by giving geometric interpretations as far as possible.

Prerequisites: A thorough knowledge of linear algebra. The third year advanced algebra courses would be a distinct advantage.

Assessment: Assignments, and a written examination, to be held at the end of semester 2.

References: David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag 1994.

M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley 1969.

N. Bourbaki, *Algèbre commutative, Ch. 8–9*, Masson, NY 1983.

S. Lang, *Algebra (3rd edn)*, Addison-Wesley 1993. (For background in algebra).

Course content: The course will include topics from the following.

1. Basic ideas. Commutative algebras over a field; affine varieties and commutative algebras; examples. Noetherian rings, Hilbert basis theorem; ideals, prime and primary ideals, decomposition theory. Localisation.

2. Integral extensions, the Nulstellensatz, geometric consequences. Definition of Spec(R). Noether normalisation. Filtered and graded rings; completions; flatness. Homological functors: Ext and Tor.

3. Dimension theory; Poincaré or Hilbert series; morphisms of varieties and their fibres.

4. Tangent and cotangent spaces; local properties of morphisms. Examples: deteminantal varieties, group schemes,

Functional Analysis —J. Parkinson

Modern functional analysis is the study of infinite dimensional vector spaces, and linear transformations between such spaces. Thus it can be thought of as linear algebra in an infinite dimensional setting.

To get the theory going we add some topology to the vector space: A *normed vector space* is a vector space with a concept of distance, and a complete normed vector space is called a *Banach space*. If the norm of the Banach space comes from an inner product (like the dot product in $\mathbb{R}^n$) then the Banach space is called a *Hilbert space*. These spaces should be thought of as the direct generalisation of $\mathbb{R}^n$ into the infinite dimensional setting, because they share many geometric properties in common with the former spaces.

We will begin by studying the geometric and structural properties of Banach and Hilbert spaces, illustrating the theory with a wide range of examples. We prove the *Stone-Weierstrass Theorem*, which has applications to the approximation of continuous functions on compact sets by polynomials and trigonometric polynomials.

We continue with a study of linear operators on Banach and Hilbert spaces, and prove three fundamental theorems of functional analysis: The *Open Mapping Theorem*, the *Principle of Uniform Boundedness*, and the *Hahn-Banach Theorem*. We will illustrate the importance of these theorems by listing many immediate corollaries.

We will also consider *spectral theory* for operators between Banach and Hilbert spaces. This is the generalisation of the notion of eigenvalues and eigenvectors, and proves to be extremely important in many applications, such as the modern theory of partial differential equations, mathematical physics, and probability theory.

Assessment:

Assignments, Quizes, and an Exam.

Prerequisites:

MATH3961 Metric Spaces (Adv)

MATH3969 Measure Theory and Fourier Analysis (Adv).

References:

A Course in Functional Analysis (J. Conway, Springer, 1990)

Functional Analysis (K. Yosida, Springer-Verlag, 1980).

Lie Algebras —A. Henderson

The theory of Lie algebras is one of the foundations of modern algebra, with applications in many other areas of mathematics and physics. This course will introduce and motivate the basic definitions of finite-dimensional Lie algebras and their representations, with particular emphasis on the general linear Lie algebra. Our goal will be the beautiful classification and description of irreducible representations, known as 'highest weight theory'. The plan is as follows:

a) Motivation: homomorphisms of general linear groups, multilinear algebra, linearization, Lie's theorem
b) Lie algebras: definition of a Lie algebra, first examples, isomorphisms
c) Basic structure: Lie subalgebras, ideals, quotients, simple Lie algebras
d) Modules: definition of a module, isomorphisms, submodules, irreducible modules, completely reducible modules
e) Theory of $\mathfrak{sl}_2$-modules: classification of irreducibles, complete reducibility
f) General theory of modules: duals, tensor products, Hom-spaces, bilinear forms, Schur's Lemma, Killing form, Casimir operators
g) Integral $\mathfrak{gl}_n$-modules: weights, highest-weight modules, irreducibility, tensor-product construction, complete reducibility
h) Further theory as time permits

Assessment:

The total mark will be made up of two assignments worth 20% each, and a final take-home exam worth 60%.

Prerequisites:

One of the appealing things about the theory of Lie algebras is that it can be approached from a relatively low base of knowledge. The only logical prerequisite is a solid understanding of linear algebra, and in particular the basic facts about matrices and eigenspaces, as in MATH2961 and MATH2968. Familiarity with representations of finite groups as in MATH3966 would be helpful but is not directly needed.

References:

Of the many books dealing with these topics, the following are closest to the spirit of the course. The first is aimed at advanced undergraduates, the others at a slightly higher level.

- *Introduction to Lie Algebras*, Karin Erdmann and Mark J. Wildon.
- *Representation Theory: A First Course*, William Fulton and Joe Harris.
- *Introduction to Lie Algebras and Representation Theory*, James E. Humphreys.

3.2. Fourth Year Courses — Semester II

Spectral theory on Banach spaces and applications to elliptic partial differential equations—N. Dancer

Spectrum of linear operators

Spectrum of compact operators

Weak derivatives and Sobolev spaces

Spectrum of linear elliptic equations

Maximum principles

This is intended as a follow on to the functional analysis course.

Assessment:

Examination 60% and two assignments worth a total of 40%

Prequisites:

Metric Spaces, first semester Honours Functional Analysis, and prefereably Lebesque Integration

References:

Martin Schechter, *Principles of Functional Analysis*, Graduate Studies in Mathematics, vol **36**, 2002, AMS.

The graded representation theory of symmetric groups—A. Mathas

The representation theory of the symmetric groups is a very rich beautiful subject which is at the heart of algebraic combinatorics and modern representation theory. In the semisimple case this subject is well-understood with the irreducible representations and their characters being known explicitly. Many fundamental questions are yet to be answered in the non-semisimple case. The aim of this course is to understand Brundan and Kleshchev's very recent and exciting discovery of a $\mathbb{Z}$-grading on the group algebra of the symmetric group to understand the representation theory of these algebras.

The course will start slowly with an introduction to the representation theory of (graded) non-semisimple algebras. In the second half of the course we will focus on the particular example of the symmetric groups. I will make the course as self-contained a possible, but knowledge of the semisimple representation theory of groups will certainly be students.

Assessment:

Two assignments worth 10-15% (to be agreed upon in the first week), with the remainder of the assessment being an exam.

References:

J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484.

The Banach-Tarski Paradox and Amenability—A. Thomas

An application of the Banach–Tarski paradox is the following bizarre statement:

A sphere can be cut into finitely many pieces which can be put back together to create two spheres, each the same size as the original one.

We will begin by making precise this sort of statement, and then state and prove the Banach–Tarski paradox. One main ingredient of the proof is the Axiom of Choice, and the other is the fact that a free group on two generators does not satisfy a property called *amenability*. Roughly speaking, a group G is amenable if there is a measure on the set of bounded functions on G that is invariant under translation by group elements. We will prove the equivalence of this definition and one of the many other formulations of amenability, involving the large-scale geometry of Cayley graphs. Using one or the other formulation, we will be able to establish the main examples of amenable and non-amenable groups, and to consider the relationship between the amenability of G and of its subgroups, quotients and extensions, culminating in a discussion of soluble groups.

Assessment:

Three assignments each worth 10% and one closed-book exam worth 70%.

Prerequisites:

This course will involve group theory, measure theory and functional analysis. Thus it is strongly recommended that students have taken:

- MATH2968 Algebra (Advanced)
- MATH3969 Measure Theory and Fourier Analysis (Advanced)
- the first semester of Honours Functional Analysis

Course outline:

Overview, history and motivation (1 lecture)

The Banach-Tarski paradox (5 lectures)

- Definition of paradoxical decompositions of sets and of paradoxical groups. Role of the Axiom of Choice.
- The free group $\mathbb{F}_2$ on two generators, proof that it is paradoxical.
- Review of $O(3)$, $SO(3)$ and their action on $\mathbb{R}^3$. Proof that $SO(3)$ has an $\mathbb{F}_2$ subgroup.
- Statement and proof of the Banach–Tarski paradox.

Invariant means (6 lectures)

- Review of measure theory, bounded linear functionals
- Locally compact groups, compact groups and discrete groups: definitions and examples
- Invariant means for discrete groups
- Application: amenability of finite groups
- Invariant means for locally compact groups
- Application: amenability of compact groups. Examples.

Følner sequences (5 lectures)

- Cayley graphs and quasi-isometries, examples
- Definition of Følner sequences

- Equivalence of amenability and the existence of a Følner sequence
- Application: non-amenability of $\mathbb{F}_2$

Amenability and group theory (6 lectures)

- Review of normal subgroups, quotient groups and group extensions
- Proof that the class of amenable groups is closed under taking subgroups, quotients and group extensions
- Application: further examples of groups that are and are not amenable
- The so-called von Neumann conjecture, and the Tarski monster as a counterexample
- Definition and examples of soluble groups
- Application: soluble groups are amenable. Implications for the geometry of soluble groups.

Summary and review (1 lecture)

References:

1. B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)*, Cambridge University Press, Cambridge, 2008.

2. A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Birkhäuser, Basel, 1994.

3. A.T. Paterson, *Amenability*, American Mathematical Society, Providence, 1988.

4. V. Runde, *Lectures on Amenability*, Springer-Verlag, Berlin, 2002.

5. R.J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.

A man ought to read just as the inclination leads him; for what he reads as a task will do him little good
Samuel Johnson

CHAPTER 4

The essay

4.1. Introduction

The essay project has several objectives. First and foremost, it is intended to provide an essentially open-ended framework whereby you may pursue, develop and discover your interests in mathematics unencumbered by syllabus and the prospect of eventual written examination. Basic to this process is the use of the library and communication with others, most especially your supervisor. The writing of the essay is a most valuable part of the project. The very act of writing is an invaluable aid to comprehension. A good essay should be carefully organised, clear, readable by others, laid out well, properly referenced and convey the essential ideas. Attainment of such writing skills is of great benefit whether or not you elect to stay in mathematics.

One point should, perhaps, be emphasized: the essay project is *not* intended to be a contribution to original research; however, the essay must clearly demonstrate that you understand and have mastered the material. Originality in presentation or view in the essay is required.

4.2. Choosing a supervisor and topic

Choosing a supervisor and topic are the first two things that you should do, and are really not two choices, but one. It is recommended that you begin in the long vacation (preceding your fourth year) by seeking out members of staff and asking them about their interests and topics they would be keen on supervising. (See also Chapter 5 below). It is a good idea to ask them about their particular method of supervising and other questions important to you. Do not feel you must settle for the first person you talk to!

All staff members, lecturer and above, are potential supervisors.

There is not necessarily any correlation between supervising style and lecturing style. Also, the subject a lecturer taught you may not be their real area of interest. You should try to decide on a supervisor and topic *before* the start of first semester. Most Department members will be available during the last two weeks of the long vacation and so, if you have not arranged a topic and supervisor at the beginning of the long vacation, you will probably have to organise your supervisor and topic during these last two weeks.

Changes in supervisor and/or topic are possible during the year (the earlier the better!). If you do change supervisors then you must notify the fourth year coordinator.

It is a good idea to have a provisional topic and supervisor in mind at the beginning of the long vacation. Your potential supervisor can then suggest some reading over the vacation and, if you have second thoughts about the topic or supervisor, it is then easy to change before the first semester starts.

4.3. Essay content and format

The essay must start with an introduction describing the objective and contents of the essay. The essay may end with a summary or conclusion; however, this is optional. Should you wish to make any acknowledgements, they should appear on a separate page, following the introduction.

You should aim at the best scholarly standards in providing bibliographic references. In particular, clear references to cited works should be made, where appropriate, throughout the text. Furthermore, it is not acceptable to base large portions of your essay on the existing literature and whenever part of your essay closely follows one of your sources this must be *explicitly acknowledged* in the text. References should not appear in the bibliography unless they are referred to in the text. For the format of the references see the appendix.

The essay should be clear, coherent, self contained and something that others (your fellow students and other non-specialists in the topic) can read with profit. The essay should not exceed (the equivalent of) 60 pages one and a half spaced type of normal $\TeX$ font size (i.e., as on this sheet). About 40–50 pages would normally be acceptable. Students are asked to try to keep their essays within these limits; overly long essay may be penalized. Supervisors should advise their students accordingly.

Take pains over style: especially clarity, precision and grammar. Aim at readability for the non-specialist. Avoid starting sentences with symbols. Aim for succinct statements of theorems and lemmas. Break up long proofs into lemmas. Cross reference previous results and notation as this markedly improves readability.

Finally, the essay must be typed or printed and prepared in accordance with the instructions listed in the appendix. These days most Fourth Year students prepare their essays using a word-processing program such as LaTeX.

4.4. Submission of essay, assessment, corrections

Three copies of the essay should be given to the Course Coordinator for marking *not later than the second Friday following the mid-semester break in second semester*.

Students should be aware that at least two weeks is set aside for marking of the essay and if the essay is not marked before the examiners meeting at the end of November then it will **not** count towards the final mark obtained in Fourth Year.

If, during the year, illness or other personal circumstances seem likely to increase the probability of late submission of the essay, such matters should be reported to the Course Coordinator through your supervisor. *Do not present such evidence at the last minute!*

Each essay will be read independently by at least two members of the Department. (The number of readers will depend on the staff available). One of the readers may or may not be the candidate's supervisor. The markers may suggest corrections should be made to the manuscript. If corrections are required, a final corrected copy of the essay should be given to the Course Coordinator as a Departmental record of the essay. If no corrections are required, one of the markers' copies will normally be kept by the department and the remaining two copies returned to the candidate.

4.5. Time management and progress reports

At the end of the first semester you should write a summary (approximately one page in length) of your essay project and progress and give this to the Course Coordinator. Here are some rough guidelines and deadlines:-

- Select supervisor and topic - Before beginning of first semester
- Reading, discussion and understanding - first semester
- Start work on first draft - beginning of second semester

- Final proofreading - mid-semester break

The essay should be submitted by the second Friday following the second semester break.

Do not underestimate the time it takes you to do the actual writing. Often it is not until you start writing that you will settle on a final view, or realise that you have misunderstood a particular part of the theory. Allow yourself sufficient time both for the typing and proof reading of the manuscript.

4.6. Your supervisor

To get the most benefit from the course, you should work closely with your supervisor. To this end, you may set up a regular hour each week to meet and discuss progress and problems with your essay project. Alternatively, you might come to some more informal arrangement.

You can expect your supervisor to:

- Help you select or modify your topic;
- Direct you to useful sources on your topic;
- Explain difficult points;
- Provide feedback on whether you are going in the right direction;
- Advise you on other course matters.

4.7. Use of the library

The Mathematics Library is located in Jane Foss Russell Building G02 (Hours (subject to change): 9:00 - 5:00, Monday to Friday). It is among the most complete mathematical libraries in Australia.
The most useful source is *Mathematical Reviews*, which is available on–line at `http://ams.rice.edu/mathscinet`. *Mathematical Reviews* is updated monthly and contains reviews of a paragraph or so, describing the contents of recent books and research articles, grouped together by research area. It also has extensive subject and author search facility. Every search for information on what has been done in a given area should begin with *Mathematical Reviews*. Your supervisor can help you learn how to effectively use reference sources like *Mathematical Reviews* and, together with the Librarian, will gladly assist you with any other problem concerning library use.

It is a worthwhile and enjoyable habit to glance at each new journal and book to see if it contains a relevant or interesting article.

A magazine of general interest that frequently has excellent articles on developments in contemporary mathematics is the monthly *Mathematical Intelligencer*. Two other journals that usually carry expository articles of high quality are the *American Mathematical Monthly* and *L'Enseignement Mathématique*. (The majority of the articles in the latter journal are in English).

> The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact.
>
> A. Whitehead, *Science and the Modern World*

CHAPTER 5

Sample essay topics

Here are some topics or areas of interest suggested by members of the Department for 2011. Please note that this list is *not* intended to be complete, the topics suggested are perhaps best regarded as a guide to the likely interests of the proposer, and other staff members are willing to act as essay supervisors. The topics are grouped according to the Research Group to which each staff member belongs.

5.1. Algebra Research Group

A/Prof David Easdown — Carslaw 619.

A/Prof David Easdown will be on study leave in Semester 2 of 2011.

Dr Anthony Henderson — Carslaw 805.

I would be happy to supervise an Honours essay in some area of representation theory or related parts of combinatorics and geometry. The following sample topics could be tailored to the student's prior knowledge.

a) *McKay correspondence.* This is a description of the finite subgroups of the unit quaternions, essentially describing rotations in three-dimensional space. The combinatorial data involved – simply-laced Dynkin diagrams – occur in many different contexts, including reflection groups, Lie algebras, and the theory of singularities.

b) *Representations of quivers.* This theory systematizes linear algebra problems such as classifying pairs of linear maps between vector spaces of a fixed dimension. A highlight is Gabriel's Theorem, which reveals the connection with reflection groups and Lie algebras.

c) *Partition combinatorics.* A partition is merely a finite weakly decreasing sequence of positive integers. Many sets of interest in representation theory are in bijection with the set of partitions (or collections of partitions) satisfying certain properties.

d) *Finite groups of Lie type.* The leading example of this class is the general linear group over a finite field, whose character table was computed by Green in one of the seminal papers of the second half of the twentieth century.

e) *Schubert varieties.* The study of these varieties, their intersections and singularities, is of great importance in algebraic geometry. It is now deeply embedded in representation theory as well, via the ubiquitous Kazhdan-Lusztig polynomials.

A/Prof. Bob Howlett — Carslaw 709.

I would be happy to supervise a fourth year essay on any topic related to my areas of research expertise. These are all various branches of algebra, involving group theory and/or representation theory. The four main areas are as follows.

a) *Representation theory of finite groups of Lie type* Lie theory encompasses a large variety of topics that are in one way or another related to the work of the 19th century mathematician Sophus Lie on continuous transformation groups. The central objects of study in this area are these days known as *Lie algebras*, although in the 19th century they were called *infinitesimal groups*. It turns out that there are seven types of simple Lie algebras, imaginatively called types A, B, C, D, E, F and G. As a finite group theorist, I am interested in finite groups of these same types; they are basically obtained by replacing the field of complex numbers by various finite fields. These finite groups of Lie type include matrix groups (such as the general linear group, orthogonal groups and symplectic groups) as well as the slightly more mysterious groups obtained from the so-called exceptional Lie algebras (types E, F and G).
b) *Finite Coxeter groups and Iwahori-Hecke algebras* Associated with each simple Lie algebra over the complex field is a finite group known as the *Weyl group* of the Lie algebra. It turns out that these Weyl groups can be realized as finite groups generated by reflections acting on Euclidean space. The Canadian mathematician H. S. M. Coxeter studied and classified finite Euclidean reflection groups,

and these are now known as Coxeter groups in his honour. They include a few groups that are not Weyl groups. There is a beautiful and extensive theory of Coxeter groups, and it seems that almost all questions in Lie theory lead back to questions about Coxeter groups. It has also been discovered that each Coxeter group has a "deformation", or "q-analogue", known as an Iwahori-Hecke algebra, and it is profitable to study these at the same time as one studies Coxeter groups.

c) *Infinite Coxeter groups* Although finite Euclidean reflection groups provided the original motivation for studying Coxeter groups, there are also infinite Coxeter groups, and they are becoming ever more important in generalizations and extensions of Lie theory. Most of my work in recent years has been concerned with advancing the theory of infinite Coxeter groups, attempting to find the correct ways to extend the well-established theory of finite Coxeter groups.

d) *Representation theory of finite soluble groups* Unlike topics (a), (b) and (c) above, this topic is not derived from Lie theory. The basic problem in representation theory of finite groups is to find ways of constructing and describing the irreducible representations of any given finite group G. Here a "representation" of G means a homomorphism from G to a group of linear operators on a vector space, and the representation in "irreducible" if there are no non-trivial proper invariant subspaces. It is hard to find general methods that apply to all finite groups, but for soluble groups a theory can be developed. The crucial fact is that if G is soluble then G has a series of normal subgroups $\{1\} = G_0 < G_1 < \cdots < G_n = G$ such that for all i the quotient group G_i/G_{i-1} is Abelian. There are theorems that describe the relationships between the irreducible representations of a group G and those of N and G/N, where N is a normal subgroup of G. This topic is known as *Clifford theory*. Repeated use of Clifford theory provides a method for proving many different theorems about representations of soluble groups.

For more information, see Dr Howlett's web page:
`http:\\www.maths.usyd.edu.au\u\bobh\`.

Professor Gus Lehrer — Carslaw 813.

There are several possible topics, which come from two basic themes.

a) The first theme is that of *symmetries of algebraic varieties*; the solution sets of polynomial equations often have symmetries (for example when the defining polynomials are permuted by a group). This leads to actions on the homology of interesting topological spaces, such as discriminant varieties and toric varieties, the latter being defined by monomials associated with polyhedral cones. Topics available here include: computing actions via rational points; geometry of certain classical algebraic varieties; topological and geometric problems arising from reflection groups.
b) The second theme is the *representation theory of associative algebras*. There are many semisimple algebras (with easily described representations) which may be deformed into non-semisimple ones by variation of parameters. These occur in algebra (Brauer, Hecke algebras), in mathematical physics (Temperley-Lieb algebras) and toplology (BMW algebras, etc). "Cellular theory" allows one to reduce deep questions about these deformations to (usually hard) problems in linear algebra. There are several possibilities for essay topics in this area.

Professor Andrew Mathas — Carslaw 718.

I would be happy to supervise a fourth year essay on any topic in representations theory, or combinatorics. My main research interests are the representation theory of the symmetric groups and related algebras (such as Hecke algebra, Ariki Koike algebras, Schur algebras, general linear groups, Brauer algebras, Solomon's descent algebras...), with an emphasis of the non-semisimple case—which is where things start to get interesting, and more difficult!

Possible topics include:

a) *The modular representation theory of finite groups* In characteristic zero every representation of a finite group can be decomposed, in a unique way, as a direct sum of irreducible representations. For fields of positive characteristic this is no longer the case, but nevertheless the number of times that a given irreducible module can arise as a composition factor of a representation is uniquely determined. Possible projects in this area range from classifying the number of irreducible representations of a finite group, to studying the Brauer and Green correspondences.

b) *Representations of symmetric groups* The representation theory of the symmetric group is a rich and beautiful subject which involves a lot of algebra and combinatorics. Possible projects here include character formulae, classifying homomorphisms, computing decomposition matrices, Murphy operators, the Jantzen sum formula...

c) *Brauer algebras* The Brauer algebras arise naturally from th= e representation theory of the symplectic and orthogonal groups, but they can also be understood from a purely combinatorial viewpoint in terms of = a "diagram calculus". Possible topics in this area include character formulae, classifying semisimplicity, branching theorems,...

d) *Seminormal forms* For many algebras it is possible to give "nice" generating matrices for the irreducible representations in the semisimple case. These explicit matrix representations are called seminormal forms. The study of the seminormal forms, and the resulting character formulae, for one or more algebras would make an interesting essay topic.

A/Prof Alex Molev — Carslaw 707.

a) *Symmetric functions* The theory of symmetric functions is a classical area of algebraic combinatorics. It is closely related to geometry of algebraic varieties, representation theory of finite groups and Lie algebras, and has a number of applications in mathematical physics. In this project we will study multiparameter symmetric functions from the combinatorial and algebraic viewpoints.

b) *Lie algebras and quantum groups*. The study of quantum groups has occupied a central stage in mathematics research for the past two decades. The groundwork for this field was laid in the mid-80s. 'Quantum groups' refer to a range of Hopf algebras that are deformations (quantizations) of either algebras of functions on groups, or universal enveloping algebras. The aim of the project is to study families of quantum groups associated with the classical Lie algebras.

Dr Bill Palmer — Carslaw 521.

Possible topics include:

a) *Bhaskar Rao designs* A Bhaskar Rao design is a matrix with group element entries, which has certain very interesting properties. There are many open questions concerning the existence of Bhaskar Rao designs over finite groups. I am currently working on the existence problem for Bhaskar Rao designs over dihedral groups.
b) *Latin squares* J. Dénes and A. D. Keedwell's recent book *Latin Squares* (North-Holland, 1991), reveals some interesting topics concerning the existence of Latin squares and the connections between these squares and codes and non–associative binary systems.
c) *Complete mappings, transversals and Latin squares*
d) A *transversal* of the Cayley table of a group of order n is a set of n cells, one in each row, one in each column, such that no two of the cells contain the same symbol. A *complete mapping* of a group is equivalent to a transversal in a Cayley table of the group. For some large families of finite groups, complete mappings are known to exist. A long-standing conjecture of Hall and Paige: *A finite group G whose Sylow $2-$subgroup is non-cyclic possesses a complete mapping* is the subject of current interest. An interested student would find the book: *Orthomorphism Graphs of Groups*, Anthony B. Evans, Springer-Verlag, 1992, and its review in *Mathematics Reviews* well worth reading.
e) *Designs, in general* The recent 1996 survey: *CRC Handbook of Combinatorial Designs* by Charles J. Colbourn and Jeffrey H. Dinitz, contains excellent up-to-date summaries of exciting advances in research in design theory. An interested student would be sure to find a worthwhile essay topic in these 748 pages.

Professor Ruibin Zhang — Carslaw 722.

a) *Quantum groups*

   Quantum groups are 'quantized versions of universal enveloping algebras of Lie algebras. They originated from the study of the Yang-Baxter equation in physics in the 1980s, and have had very significant impact on many branches of mathematics and physics in recent years. Research on quantum groups is very active, and the subject is rapidly developing. M. Jimbos original paper, A q-analogue of U(gl(N+1)), Hecke algebra, and the Yang-Baxter equation, Lett. Math. Phys. 11 (1986), 247252, is a good place to get a taste of the subject.
b) *Infinite dimensional Lie algebras*

Infinite dimensional Lie algebras play a key role in conformal field theory and the theory of strings. Typically the Hilbert space of a physical system forms a positive energy module over the Virasoro or a Kac-Moody algebra, where the energy operator is some special element of the algebra. The study of the states and the energy spectrum of the physical system thus may be treated algebraically within the representation theory of these infinite dimensional Lie algebras. Thesis topics in this area involve studying such representations of the Virasoro and Kac-Moody algebras that are most commonly used in physics. A very readable paper with a physics flavour is: Goddard, P.; Olive, D.I., Kac-Moody and Virasoro algebras in relation to quantum physics. Internat. J. Modern Phys. A 1 (1986), 303–414.

c) *Lie superalgebras and supersymmetry*

Supersymmetry is a basic principle which ensures that the fundamental laws of physics are the same for bosons and for fermions. It has permeated many areas of pure mathematics in recent years, leading to deep results such as the Seiberg-Witten theory and mirror symmetry. The algebraic perspective of supersymmetry is the theory of Lie superalgebras, which came to existence in the late 1970s, and is still actively studied today. We shall develop the structure and representations of Lie superalgebras such as the general linear superalgebra and its subalgebras. Preliminary reading: Kac, V. G. Lie superalgebras. Advances in Math. 26 (1977), 8–96.

d) *Quantum field theory*

Quantum field theory is the conceptual framework for formulating fundamental laws of physics and answering questions about the structure of the physical world. It provides the mathematical means for studying quantum systems of infinitely many degrees of freedom, and making definite predictions which can be tested at experimental facilities like the LHC at CERN. An authoritative and readable introduction to the area is Steven Weinberg, The Quantum Theory of Fields I: Foundations (Cambridge University Press, 1995). Current reach activities in the area largely focus on the development of a quantum field theory of gravity which recovers Einsteins general relativity as the classical limit. Such a theory is necessary for understanding the structure of spacetime at small scale (the Planck scale), when the description of spacetime in terms of pseudo-Riemannian manifolds is no longer adequate.

5.2. Analysis Research Group

Dr Donald Cartwright — Carslaw 638.

a) *Analysis on trees* A homogeneous tree T is an infinite graph, in which there are no “loops", and in which each vertex has the same number of neighbours. Because T is highly symmetric, it has a very large group of automorphisms, and many nice representations of this group can be defined in a natural way. Research in this area has been very active over the last decade, but the subject is quite accessible. Preliminary reading: *Harmonic Analysis and Representation Theory for Groups acting on Homogeneous Trees*, by A. Figá-Talamanca and C. Nebbia (Cambridge University Press, 1991).

b) *Analysis on the hyperbolic plane* The “hyperbolic plane" is the unit disc in the ordinary plane, endowed with a different idea of distance: *hyperbolic distance*. It is an example of a “symmetric space", which means roughly that there is a large group of (hyperbolic-)distance preserving transformations acting on it. These give rise to beautiful tesselations of the disc which inspired many designs by the artist Escher. An interplay of geometry, complex analysis, number theory and a little bit of group theory. Preliminary reading: Chapter III of *Harmonic Analysis on Symmetric Spaces and Applications. I* by A. Terras, Springer-Verlag (1985).

c) *Random walks on groups* The classical example is of a point moving on the group $\mathbb{Z}$ of integers, at each time hopping one to the right or one to the left with probability 1/2. One studies what happens "in the long run". Considering the natural extension to more general groups leads to an interesting interplay between analysis and group theory, which has received much attention in recent years. Some background in probability and measure theory would be useful, but not essential. Preliminary reading: *Random Walks and Electric Networks* by P. Doyle and L. Snell, Mathematical Association of America (1984).

Dr Adrian Nelson — Carslaw 526.

Dr Adrian Nelson will be on study leave in the Semester 2 of 2011.

Dr James Parkinson — Carslaw 614.

I would be happy to supervise fourth year students on various aspects of combinatorial representation theory, probability theory, and harmonic analysis on graphs and groups. Some specific ideas include:

(a) *Representation theory and random walks.* The classical example here is that of a random walk on a d-dimensional lattice. But the underlying object can be just about anything you like. If the random walk is on a group, then the representation theory of the group can be used to give very detailed information about the random walk. This is Fourier analysis on groups, and it becomes very interesting when the group is nonabelian and infinite. This project could, for example, apply representation theory of groups of Lie type and their associated Hecke algebras to study random walks on beautiful geometric objects called 'buildings'.

(b) *Affine Kac-Moody algebras and the Macdonald identities*

Kac-Moody algebras are a remarkable generalisation of the finite dimensional semi-simple Lie algebras into the infinite dimensional setting. These algebras are divided into three categories: *Finite dimensional*, *affine*, and *hyperbolic*.

The finite dimensional Kac-Moody algebras are precisely the finite dimensional semi-simple Lie algebras. These algebras are well understood. The affine Kac-Moody algebras also have a well developed theory, because these algebras can be constructed from the finite dimensional algebras. Understanding this connection would form the first part of the project. One could then move on to study the representation theory of affine Kac-Moody algebras. There is some truly beautiful mathematics here: The Weyl denominator formula generalises, and one recovers the intriguing *Macdonald identities*.

(c) *The symmetric group, card shuffling, and cut-off phenomenon.*

Many systems display a *cut-off phenomenon*: After $k-1$ iterations the system is still relatively ordered, yet after k iterations the system is close to random. A celebrated example is the card shuffling theorem of Dave Bayer and Persi Diaconis: *7 suffles are required to sufficiently randomise a deck of 52 cards*.

This theorem has had real applications to how casinos deal their cards (it was previously thought that 3 or 4 shuffles would be enough). The proof relies on a careful study of *Solomon's descent algebra* and the character theory of the symmetric group. Cut-off phenomena occur in many other places. For example very recently Jason

Fulman defined an intriguing random walk on the set of irreducible representations of a finite group, and proved cut-off phenomena for certain groups. A close study of the works of Bayer, Diaconis and Fulman (with possible extensions to other examples) would make for a fascinating project.

(d) *Harmonic analysis on graphs and groups*

Harmonic analysis is the abstract study of Fourier analysis. The classical setting is Fourier analysis on $\mathbb{R}^d$ or $\mathbb{Z}^d$, but there are far reaching generalisations to other groups and graphs. Examples where a rather explicit theory exists include lattices, trees, free groups, Coxeter groups, real Lie groups, and p-adic Lie groups. A detailed study of harmonic analysis on homogeneous trees would be a good introduction to this theory. There are nice applications to the study of random walks on trees.

Dr Anne Thomas — Carslaw 615.

I would be happy to supervise an Honours essay on any topic related to my research interests, which include geometric group theory, rigidity and lattices in locally compact groups. Some sample topics are as follows.

a) *Hyperbolic groups.* One of the fundamental ideas in geometric group theory is to study finitely generated groups via the geometry of their associated Cayley graphs. A hyperbolic group is a group whose Cayley graph is “negatively curved" in an appropriate sense. Hyperbolic groups share many properties with fundamental groups of compact surfaces of genus at least 2. Possible directions include algorithmic properties, boundaries, separability properties and the surface subgroup conjecture for hyperbolic groups. Suggested reading: Bridson and Haefliger, *Metric Spaces of Non-positive Curvature*.

b) *Bass-Serre theory.* Group actions on trees may be encoded using the combinatorial data of a graph of groups. This beautiful theory has many applications to the study of group splittings, low-dimensional topology and lattices in Lie groups over nonarchimedean fields. Suggested reading: Serre, *Trees*.

c) *Expanders and Property (T).* Expanders are highly-connected sparse graphs which have many applications in computer science. The first explicit construction of expanders, by Margulis in 1973, made use of a representation-theoretic property, Kazhdan’s Property (T). This topic brings together group theory, combinatorics, geometry and analysis. Suggested reading: Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*.

d) *Coxeter groups and nonpositive curvature.* Coxeter groups, which are generated by reflections, are ubiquitous in mathematics. Infinite Coxeter groups may be studied via their action on a contractible cell complex, the Davis complex, which satisfies an important combinatorial nonpositive curvature condition called the CAT(0) inequality. Suggested reading: Davis, *The Geometry and Topology of Coxeter Groups*.

5.3. Computational Algebra Research Group

Professor John Cannon — Carslaw 618.

a) *Computational Number Theory*
   For example:
   - Primality testing and factorization
   - Constructive algebraic number theory
   - Computation of Galois groups

b) *Computational Group Theory*
   - Algorithmic methods for finitely presented groups
   - Algorithmic methods for permutation groups
   - Computational representation theory
   - Constructive invariant theory

c) *Computational Differential Algebra* For example:
   - The Risch algorithm for indefinite integration

5.4. Geometry Research Group

Dr Emma Carberry—Carslaw 723.

My primary research areas are differential geometry and integrable systems, although I also often use methods from algebraic geometry in my work. I have listed some specific areas below in which I would be happy to supervise an essay; if you have other geometric interests please feel free to contact me for further ideas.

Minimal Surfaces : Physically, minimal surfaces model soap films: they locally solve the problem of finding the least area surface with a given boundary. They have been extensively studied and have a rich theory, with many interesting examples and generalisations.

They are an active area of current research. There are various possibilities here for a project; ranging from the very explicit (some of the most exciting research here involves finding new examples), to the more theoretical. David Hoffman's expository article "The computer-aided discovery of new embedded minimal surfaces" in Math Intelligencier, 9 (1987), no 3 and Robert Osserman's book " A Survey of Minimal Surfaces" are good places in which to get a feel for this area.

Calibrations : The notion of a calibrated geometry was introduced in a seminal paper by Harvey and Lawson 25 years ago. In these geometries, one studies special submanifolds which are globally area minimising (this is much stronger than the local condition that characterises minimal surfaces). The first non-classical example is special Lagrangian geometry, which plays an important part in mirror symmetry and is currently a hot research area. One can also use the octonions to define three more calibrated geometries, termed exceptional geometries due to their relationship with exceptional Lie groups. This area requires some background in differential geometry, such as that provided by MATH 3968.

Spectral Curves : There is an important class of differential equations which can be written in a particularly simple form, called Lax form. For example, the equations describing a minimal surface in a compact Lie group or symmetric space can be written in this form. It is a beautiful fact that solutions to such differential equations on the complex plane (satisfying a finiteness condition) are in one-to-one correspondence with purely algebro-geometric data, consisting of an algebraic curve and a line bundle. The curve is called a spectral curve, and this correspondence gives one powerful new tools with which to attack the original geometric problem. This area requires some background in complex analysis, such as that provided by MATH 3964. References include Philip Griffiths' article ",Linearizing flows and a cohomological interpretation of Lax equations" American Journal of mathematics, 107 (1985), no 6, 1445–1484 (1986). and the section by Hitchin in the book " Integrable Systems - Twistors, Loop Groups, and Riemann Surfaces" by Hitchin, Segal and Ward.

Quaternionic Holomorphic Geometry : When studying the geometry of surfaces, one usually works locally as there are few global tools available. A couple of years ago it was observed that surfaces in S^4 could be studied more globally, using quaternionic analogs of standard complex analytic results. The quaternions enter the picture since S^4 is isomorphic to the quaternionic projective line, and one can use these tools to study surfaces in $\mathbb{R}^3$ simply by embedding $\mathbb{R}^3$ in S^4. This new theory is being used to study conformal immersions of surfaces, and in particular to attack the Willmore conjecture. A good reference is the book "Conformal geometry of surfaces in S^4 and quaternions by Burstall, Ferus,Leschke, Pedit and Pinkall, available online at front.math.ucdavis.edu. Some background in complex analysis and differential geometry would be needed for this project.

Curves and their Jacobians : Algebraic curves (smooth algebraic curves are also called compact Riemann surfaces) and line bundles on them are utilised in many areas of mathematics. Important theorems include the Riemann-Roch theorem and the Abel-Jacobi theorem. A good reference is Philip Griffiths' book " An Introduction to Algebraic Curves".

Dr Jonathan Hillman — Carslaw 617.

Dr. Hillman will not be available for supervision in 2011.

Dr Laurentiu Paunescu — Carslaw 721.

I am interested in the applications of singularity theory to differential equations, and in using the combinatorics of Toric Modifications in investigating the equisingularity problem. My main research interests are:

a) *Singularities of complex and real analytic functions*
b) *Stratified Morse theory*
c) *Toric resolution of singularities*

Dr Zhou Zhang — Carslaw 620.

My primary research interest lies in differential geometry. Techniques from the theories of partial differential equation and several complex variables are frequently called for. The topics under consideration ofter have strong algebraic geometry background. In the following, I list a few topics suitable for an honour essay. If you already have other topics in mind, I'll more than happy to have some discussion and provide suggestions if you wish.

a) **Futaki Invariant**: the goal is to look at some basic facts regarding this very important invariant in the study of Einstein metric. The definition itself would serve as a good topic to introduce fundamentals in complex differential geometry.

b) **Ricci Flow and Maximum Principle**: Ricci flow and the complex version of it, Kähler-Ricci flow are a topic of great interest in recent years. Maximum Principle is, in principle, a very simple-minded tool in the study of differential equations. We focus on Hamilton's Tensor Maximum Principle and provide some taste of how something so intuitive can go such a long way.
c) **Introduction to Algebraic Curves**: here I am using the title of the book by Phillip A. Griffiths. It provides a few very good topics to work on for beginners in algebraic geometry, for example, the Riemann-Roch Theorem in Chapter III.
d) **V-solition and Kähler-Ricci Flow**: the notion is V-soliton is only introduced very recently in the work of La Nave and Tian. To understand its close relation with Kähler-Ricci flow, one needs to understand symplectic reduction, which is a very useful notion from symplectic geometry.

5.5. Non-Linear Analysis Research Group

Dr Florica Cîrstea — Carslaw 719.

My main research interests concern nonlinear partial differential equations (PDEs). In this area there are many important topics which can be treated using various modern approaches. I would be happy to supervise such topics from the theory of both linear and nonlinear PDEs. Some specific projects are provided below, but the students are encouraged to negotiate the topic for a best match.

I. The theory of linear PDEs relies on functional analysis and relatively easy energy estimates to prove the existence of weak solutions to various linear equations. The proper setting for the study of many linear and nonlinear PDEs via energy methods is provided by the so-called Sobolev spaces. If we require the solutions of a given PDE to be very regular, we would usually have a very hard time to find them. A more profitable approach is to consider the issue of existence of solutions separately from the smoothness (or regularity) problems. The idea is to introduce a new concept of solution (weak solution) which does not have too much smoothness so that we could establish its existence, uniqueness and continuous dependence of the given data. Various PDEs could be treated in this way and this is, possibly, the best we can do in many cases. For others, we could hope to prove that our weak solution becomes smooth enough to be deemed as a classical solution.

This leads to the issue of regularity of weak solutions, which relies on many intricate estimates.

Possible topics for supervision: investigating the solvability of uniformly elliptic, second-order PDEs, subject to prescribed boundary conditions using two essentially different techniques: energy methods within Sobolev spaces and maximum principle methods. The energy methods can be expanded to a variety of linear PDEs characterizing evolutions in time. This broadens the class of PDEs to include the heat equation and more general second-order parabolic PDEs, as well as the wave equation (and general second-order hyperbolic PDE).

II. The theory of nonlinear PDEs is far less unified in its approach compared with the linear one. Variational methods provide one of the most useful and accessible of the approaches for nolinear PDEs. Other techniques are also available for nonlinear elliptic and parabolic PDEs such as the monotonicity and fixed-points methods, as well as other devices involving the maximum principle. The study of such techniques would make interesting essay topics.

Professor Norm Dancer — Carslaw 717.

a) *Nonlinear analysis*
b) *Nonlinear ordinary differential equations*
c) *Nonlinear partial differential equations*
d) *Bifurcation theory*

Dr Daniel Daners — Carslaw 715.

Areas of interest:

a) partial differential equations (linear or nonlinear)
b) ordinary differential equations (linear or nonlinear)
c) bifurcation theory
d) analytic semigroup theory and abstract evolution equations. (This is a theory of "ordinary differential equations" in infinite dimensional spaces with applications to partial differential equations).

Please see me to negotiate a topic of your interest or for suggestions for specific projects related to the above areas.

Extraordinaire comme les mathématiques vous aident à vous connaître.

Samuel Beckett, *Molloy*

CHAPTER 6

The talk

6.1. General remarks

Before the essay is submitted at the end of Second Semester, each student gives a talk on their essay project. In past years these talks have taken place in the weeks of September leading up to the mid-semester break.

The aim of the talk is to provide training in the explanation to others of the purpose and nature of a project, within definite time limits; thirty minutes for each talk, plus fives minutes for questions.

All members of the Department, Fourth Year and postgraduate students are invited to the Fourth Year talks.

No explicit grade is given for the talk

6.2. Preparing the talk

The purpose of your talk is to convey to your *fellow students* (and the academic staff) what you are working on. They probably know very little about your essay topic; this comment may also apply to the academic staff. Do not make the talk too long or ambitious. Aim to convey the essence of your project to the audience rather than trying to impress the audience; after all, it is unlikely that you can cover the whole of your project in 30 minutes!.

The key to giving a successful mathematical talk is: "Keep it simple!" *One* idea, illustrated by one or two examples, is enough for a good talk. A special case often conveys more than a general, all-encompassing theorem. For example, to give the flavour of general fields, a detailed study of a simple, but unfamiliar field, such as $GF(9)$, might be appropriate.

Keep in mind that the audience is swept along with you and that they cannot go back to earlier stages of your talk like when they are reading an article.

You are not giving a lecture, so although some definitions may be appropriate, lengthy technical proofs should be avoided. It is also not a good idea to over-develop the theory at the expense of examples: a well-chosen example is worth ten thousand theorems. Finally, try and relate your content to other areas of mathematics or applications; this can make the talk much more interesting for the general audience.

You should aim your talk at a general mathematical audience and *avoid* directing it at the odd specialist in your topic in the audience. Thus a good talk is judged by one criterion: you have given the audience, especially your fellow Fourth Year students, a good idea of your project and its significance.

Discuss the talk with your supervisor.

Having chosen the topic for your talk, prepare a written outline. Some people write their talk out in full, while others prefer to use only a written outline and allow improvisations. As it is probably your first talk of this kind, it is advisable to do a full dress rehearsal the previous evening; so find a blackboard and an overhead project and go through the complete talk. This will help you in judging the timing of your talk properly: it takes much longer to say things than you probably realize. If you can, find a sympathetic listener to give you feedback. Your listener does not have to be mathematically literate: a good talk is almost as much about theatre and presentation as it is about mathematics.

6.3. Overhead projectors

Decide if an overhead projector is appropriate. This allows preparation of complicated figures or tables ahead of time, or the inclusion of photocopies of published material in your exposition. Beware, however, that although the speaker can by this means pass a vast amount of information before the eyes of the audience very quickly, the audience will probably not take it all in. It is important either to write clearly and in large letters and to refer explicitly to each line (say by gradually revealing line-by-line using covering paper) or, in the case of a diagram or complicated formula, to allow your audience time to absorb its detail.

If you are going to use TEX to create slides then *make sure that you use large enough fonts*; the easiest way to do this is to use a LATEX package such as `foiltex`, or a program such as powerpoint.

Life is good for only two things, discovering mathematics
and teaching mathematics.

Siméon Poisson

CHAPTER 7

Your future and mathematics

As a fourth year student you are a member of the mathematics department and you should take advantage of the facilities it offers. The University of Sydney has one of the top mathematics research departments in the country, and it ranks very highly internationally in several areas. There are also a number of prominent international (short and long term) visitors to the department who give seminar talks within the department. It pays to keep an eye on `scnews` (the School's web based bulletin board), for upcoming seminar announcements.

The academic staff, the many postdocs and the visitors to the department are all usually very happy to talk mathematics talk with interested students: all you have to do is find the courage to ask!

Fourth year students are also very welcome to join the staff and postgraduates in the use of the tea room; this can be a good place to meet other people in the department.

7.1. The colloquium and other seminars

Most Fridays during the year, a Colloquium is held at 2:00pm in either Carslaw Lecture Theatre 375 or in the 'Red Centre' UNSW. Topics vary, but the intention is to provide a one-hour talk on a subject of contemporary mathematical interest to a general audience. Fourth Year students are encouraged to attend the Colloquium and indeed are welcome to any seminar run in the Department. For a schedule of upcoming seminars, see the Department notice board on Level 7, Carslaw, or *read your e–mail*.

There are also a number of other active seminars in the department; notably, in algebra, computational algebra algebraic geometry seminar and category theory.

7.2. After fourth year, what then?

Recent graduates have found employment in a wide variety of occupations: computer related jobs, teaching (University or School), positions in insurance and finance. To find out more about where maths can take you:

`http://www.careers.usyd.edu.au`
`http://www.amsi.org.au`
`http://www.austms.org.au/Jobs`
`http://www.ice-em.org.au/careers.html` .

Here we shall just outline briefly the postgraduate degree options. For more information consult the departments web pages.

7.3. Higher degrees

A result of II-2 or better is the minimum requirement for entry into a higher degree at Sydney. However it should be noted that one should not normally contemplate continuing without a result of at least II-1. Anyone intending to undertake a higher degree should consult with the Mathematics Postgraduate Coordinator (A/Prof Andrew Mathas, Carslaw 718) as soon as possible. The usual practice is to enrol for an M. Sc in the first instance and later to convert to a Ph.D if it is desired to continue.

7.4. Scholarships and other support

Scholarships, prizes and travel grants are available both for study at Sydney and for study elsewhere. Full details can be found in the University Calendar and from the Scholarships Office (Administration Building). Intending applicants should obtain application forms from the Scholarships Office as soon as possible. *The closing dates for some scholarships can be as early as September*

If you are considering further study at an Australian University, you should apply for an Australian Postgraduate Research Award. (even for an M. Sc by coursework). For study at a university in Britain or Canada, apply for a University of Sydney travelling Scholarship and also apply to the chosen university for employment as a Graduate Assistant.

7.5. Further study in another subject

As mentioned in the introduction to this booklet, it is quite possible to do Fourth Year Pure Mathematics and then continue with a higher degree in another subject. Within Australia, prerequisites vary from Department to Department and for those intending to follow this path it is advisable to consult with the Department concerned to determine an appropriate choice of Fourth Year topics. If you are intending to continue with Postgraduate studies in another field *outside* Australia, do check prerequisites. Provided you have done third year courses in the subject at Sydney, you will *probably* not encounter significant problems over prerequisites.

It has long been an axiom of mine that the little things are infinitely the most important.
Sherlock Holmes, *A Case of Identity*

APPENDIX A

Instructions on preparing the Manuscript

Essays must be typed using LaTeX (or TeX), or a commercial word processing program such as word. Amongst professional mathematicians LaTeX has become the standard; it produces better quality output than any word processing programs program — at least when it comes to mathematics. The downside to LaTeX is that it takes some time to learn.

The fourth year coordinator will give an introduction to using TeX and LaTeX before the beginning of second semester. For those wishing to use LaTeX A/Prof Mathas has written a LaTeX *class file* which takes care of the basic layout of the essay; for information see

`http://www.maths.usyd.edu.au:/u/mathas/pm4/`

If you decide not to use this LaTeX class file, then your document must satisfy the following requirements.

a) A margin of at least 2.5cm must be left at the top, bottom, left- and right-hand side of each page. The margin is determined by the last letter or character in the longest line on the page.
b) All pages must be numbered (in a consistent way), except for the title page.
c) Avoid excessive use of footnotes. They are rarely necessary in mathematics.
d) Diagrams should be created using appropriate software; hand drawn diagrams are not acceptable.
e) Theorem Propositions, etc. should be labelled consistently throughout the document.
f) *References* A consistent scheme should be adopted. Thus, references may be numbered: [1], [2],... and referred to as such in the text. Alternatively, by author's initial: [A], [F1], [F2],...

   Sample references for the bibliography are given below:

   [10] C. Chevalley, *Theory of Lie Groups*, Princeton Univ. Press, Princeton (1946).

[15] H. Grauert, On Levi's problem and the embedding of real analytic manifolds, Ann. Math. 68 (1958), 460-472.

[28] J. N. Mather, Stratifications and mappings, in *Dynamical Systems*, Academic Press, New York (1973), 195-232.

Note: instead of using italics, the titles of books may be underlined, or placed in quotation marks. Similarly the titles of journals may be italicised or underlined.

References should be listed alphabetically.

Notes on the use of TEX and LATEX are available from Dr Howlett and the web page above.