MATHEMATICS IV HONOURS 2017
ALGEBRAIC NUMBER THEORY
ASSIGNMENT 1

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Question 1.

(1) Suppose $x \in \mathbb{C}$ is integral over $\mathbb{Z}$. Show that if the minimal polynomial of x over $\mathbb{Q}$ is written

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_n, \tag{1}$$

then $f(t)$ is irreducible in $\mathbb{Q}[t]$, and $a_i \in \mathbb{Z}$ for all i.

(2) Describe the integers in $\mathbb{Q}(\sqrt{-7})$.

(3) Write down the minimal polynomial of an arbitrary integer in the form (1).

Question 2.

(1) Define the discriminant $\Delta_{K/\mathbb{Q}}(x_1, \dots, x_n)$ of a basis $x_1, \dots, x_n$ of the algebraic number field, define the discriminant of K, and show that it is unambiguously defined.

(2) Compute the disciminant of $\mathbb{Q}(\sqrt{m})$, where m is square free and congruent to 1 mod 4.

(3) Let p be an odd prime, and let K be the cyclotomic field of degree $p-1$. Show that the discriminant of K is $(-1)^{\frac{p-1}{2}} p^{p-2}$.

Question 3.

(1) Describe the ring A of integers in $K := \mathbb{Q}(\sqrt{-5})$.

(2) Show that A is not a UFD, and therefore not a PID.

(3) Let $\mathcal{A}$ be the ideal $(2, 1+\sqrt{-5})$. Calculate the norm $N(\mathcal{A})$.

(4) Give the factorisation of principal the ideal (6) into prime ideals.

(5) Can you determine the class group of K?

Question 4. Let K be an algebraic number field of degree n over $\mathbb{Q}$, and write $\sigma_1, \dots, \sigma_n$ for the distinct $\mathbb{Q}$-isomorphisms: $K \hookrightarrow \mathbb{C}$. Write A for the ring of integers of K.

(1) If $\widetilde{K}$ is the normal closure of K (obtained by adjoining to K all conjugates of a generator of K over $\mathbb{Q}$), show that for each i, $\sigma_i : K \to \widetilde{K} \subseteq \mathbb{C}$ may be extended to an automorphism $\widetilde{\sigma}_i$ of $\widetilde{K}$.
[*Hint:* $\widetilde{K} = K(\beta)$ for some β. Compose σ_i with an appropriate map $\widetilde{K} \to \widetilde{K} \subseteq \mathbb{C}$.]

(2) Show that if $x \in A$ is such that $\widetilde{\sigma}_i(x) = x$ for $i = 1, 2, \dots, n$, then $x \in \mathbb{Z}$.

(3) Show that for each i, there is a unique permutation $\omega_i \in \mathrm{Sym}_n$ of $\{1, \dots, n\}$ such that **for all** $x \in K$, we have $\widetilde{\sigma}_i(\sigma_j(x)) = \sigma_{\omega_i(j)}(x)$.

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Question 5. Maintain the notation of Question 4.

(1) Define the discriminant of K, and show that if $\alpha_1, \dots, \alpha_n$ is a ($\mathbb{Z}$-)basis of the ring A of integers in K, then if $d = \text{discrim(K)}$, then

$$d = \det\left(\sigma_i(\alpha_j)\right)^2.$$

(2) For $\pi \in \text{Sym}_n$ (the symmetric group of degree n) let $t_\pi = \prod_{i=1}^n \sigma_i(\alpha_{\pi(i)})$. Show that in the notation of Question 4(3), we have $\widetilde{\sigma}_i(t_\pi) = t_{\omega_i^{-1}\pi}$, for any i and π.

(3) Let $T_0 = \sum_{\pi:\varepsilon(\pi)=1} t_\pi$ and $T_1 = \sum_{\pi:\varepsilon(\pi)=-1} t_\pi$, where ε is the alternating character of Sym_r (whose value at π is the sign of π). Show that

$$d = (T_0 + T_1)^2 - 4T_0T_1.$$

(4) Using Question 4(2), show that $T_0 + T_1 \in \mathbb{Z}$ and $T_0T_1 \in \mathbb{Z}$. [*Hint* Show that these are stable under each automorphism $\widetilde{\sigma}_i$.]

(5) Deduce that the discriminant $d \equiv 0, 1 (\text{mod } 4)$.

(This is Stickelberger's Theorem)

Question 6. Let $\alpha = \sqrt[3]{2}$ be the real cube root of 2, and let $K = \mathbb{Q}(\alpha)$.

(1) Describe the $\mathbb{Q}$-embeddings $K \hookrightarrow \mathbb{C}$.

(2) Show that the discriminant $\Delta_{K/\mathbb{Q}}(1, \alpha, \alpha^2) = -108$.

(3) Show that $N_{K/\mathbb{Q}}(a + b\alpha + c\alpha^2) = a^3 + 2b^3 + 4c^3 - 6abc$. Deduce that $1 + \alpha + \alpha^2$ is an integer unit in K.

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