Sydney University Mathematical Society
Problems Competition 2004

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($50 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years). Entries from fourth year students will be considered. When prizewinners are being selected, if two or more entries to a problem are essentially equal, then preference may be given to the students in the earlier year of university.

Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, September 10, 2004. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **2004 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

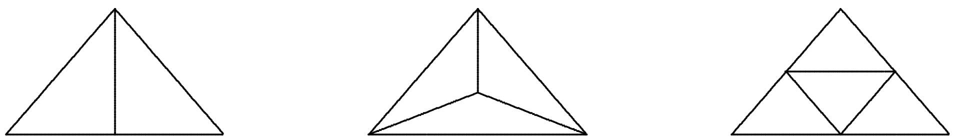
These problems will also be posted at the website

http://www.maths.usyd.edu.au/SUMS/

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. We can partition an equilateral triangle into two, three or four congruent triangles, as the following diagram shows:



Show that it is not possible to partition it into five congruent triangles.

2. For which positive integers n is $2^{n-1}+1$ divisible by n?

3. Suppose that M is an $n \times n$ matrix of 0's and 1's with the property that in each row the 1's are adjacent to each other. For example,

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Show that $\det(M)$ is either 0, 1, or -1.

4. Find a closed form expression for

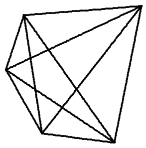
$$\sum_{k=0}^{m} \left(\frac{n!}{k!}\right)^2 \frac{1}{(n-2k)!4^k}$$

where $m = \lfloor \frac{n}{2} \rfloor$.

5. Back in 1999, the following might have made a nice SUMS problem: "Consider the sum $1-1/2+1/3-1/4+\cdots+1/1331-1/1332$, written in reduced form as m/n. Show that m is divisible by 1999." Solve this problem, and formulate a similar problem for some years in the near future.

6. Let $A \subset \mathbb{R}$. Call $x \in A$ an *interior point* of A if there is an $\epsilon > 0$ so that $(x-\epsilon, x+\epsilon) \subset A$. Let $\operatorname{Int}(A)$ denote the set of interior points of A. Let $\tilde{A}$ denote the complement of A in $\mathbb{R}$. Starting from a set, we can (i) form its interior, or (ii) take its complement. How many different sets can be obtained from a given set A by successively applying the operations (i) and (ii)?

7. Let f_n denote the number of non-overlapping regions into which the interior of a convex n-gon is divided by its diagonals. You should assume that no three diagonals meet at a point inside the n-gon. For example, the diagonals divide the following 5-gon into 11 non-overlapping regions. Find a formula for f_n.



8. Show that for any integers $m, n \geq 1$, the following expression is an integer:

$$\frac{m(m-1)}{(n+1)(mn+1)}\binom{mn+n}{n}.$$

9. A *partition* of n is a non-increasing sequence $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ of positive integers whose sum is n. Let $\mathcal{P}(n)$ be the set of partitions of n. For example, $\mathcal{P}(3) = \{(3), (2,1), (1,1,1)\}$. A *path sequence* is a doubly infinite sequence $(p_i)_{i\in\mathbb{Z}} = \ldots p_{-2}p_{-1}p_0|p_1p_2\ldots$ of 0's and 1's. We use the $|$ to mark the position of p_0. The *weight* of a path sequence $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ is defined to be the sum over the i's such that $p_i = 0$ of the number N_i of integers $j < i$ such that $p_j = 1$. For example, the following path sequences all have weight 3: $\ldots 00001|110111\ldots$, $\ldots 00010|101111\ldots$ and $\ldots 00100|011111\ldots$. Moreover, up to shift, these are all of the path sequences of weight 3.
(a) Show that there is a bijection between the partitions of n and the path sequences $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ of weight n for which

$$\sum_{i\leq 0} p_i = \sum_{i>0}(1-p_i) < \infty.$$

The examples above give such a bijection when $n = 3$.
(b) Suppose that $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ is a path sequence satisfying the condition in (a). Fix integers $j < k$ with $p_j = 1$ and $p_k = 0$. Let $\mathbf{p}'$ be the path sequence obtained by swapping p_j and p_k in $\mathbf{p}$; that is, set $p'_i = p_k$ if $i = j$, $p'_i = p_j$ if $i = k$, and set $p'_i = p_i$ otherwise. Show that $\mathbf{p}'$ also satisfies the condition in (a) and describe the partition corresponding to $\mathbf{p}'$ under your bijection (in terms of the partition corresponding to $\mathbf{p}$).

10. Let n be a positive integer. Let $a_1, a_2, \cdots, a_m$ be a partition of n (see the previous question). Represent this partition as a left-justified array of boxes, with a_1 boxes in the first row, a_2 in the second, and so on, and label the boxes with 1 and -1 in a chess-board pattern, starting with a 1 in the top-left corner. Let c be the sum of these labels. For instance, if $n = 11$ and the partition is $4, 3, 3, 1$, then $c = -1$, as one sees by summing the labels in the diagram:

1	−1	1	−1
−1	1	−1	
1	−1	1	
−1			

Prove that $n \geq c(2c-1)$, and determine when equality occurs.