Inverse design for conservation laws and Hamilton-Jacobi equations

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joint work with Carlos Esteve and Thibault Liard

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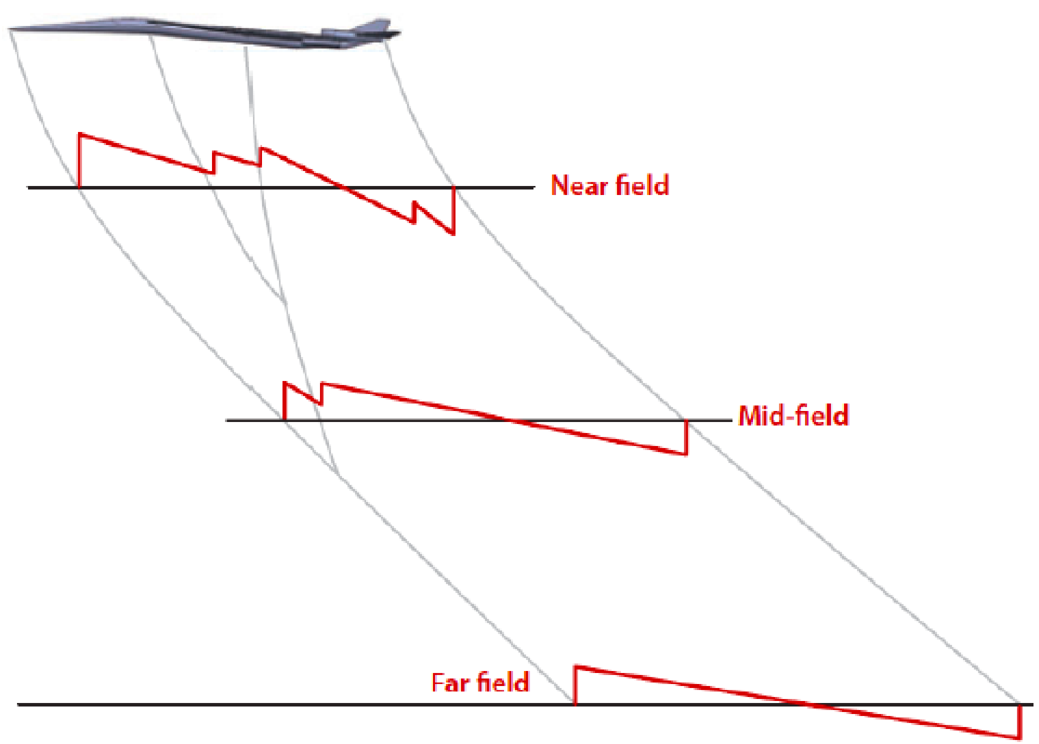
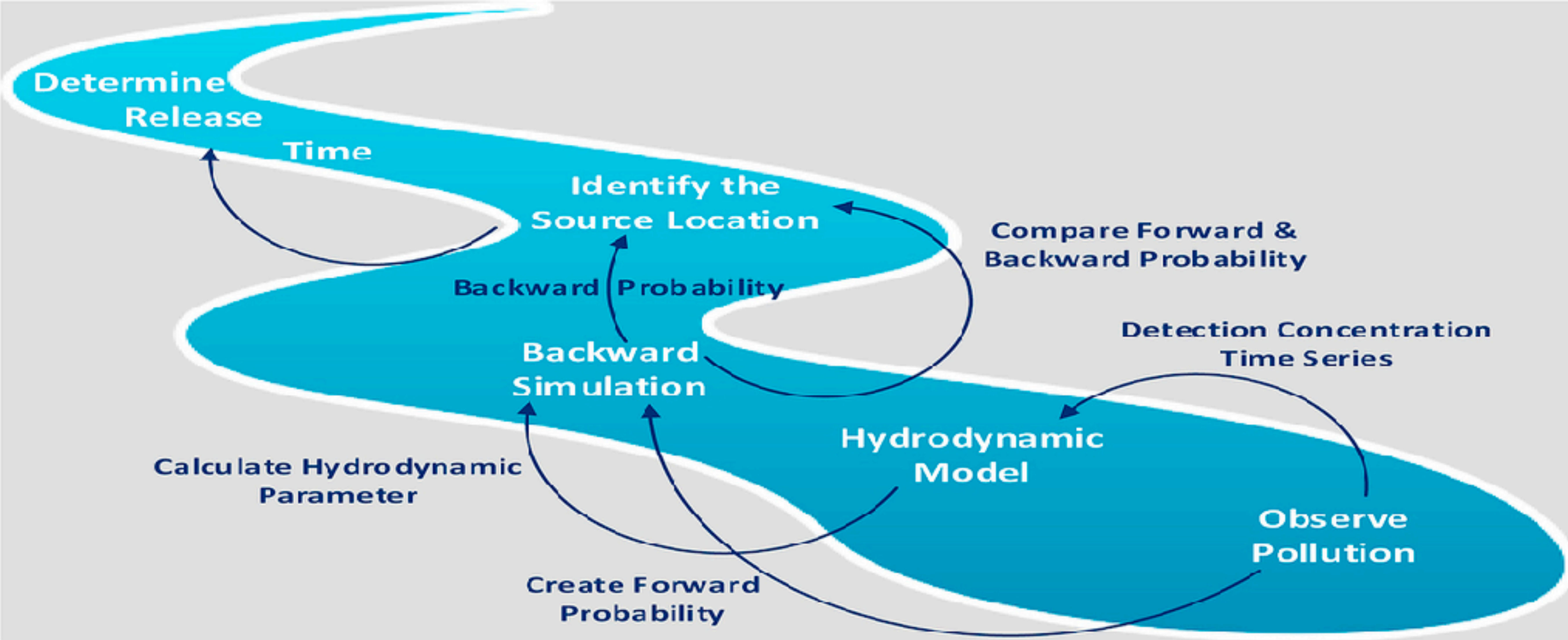
Motivation: sonic boom and supersonic airplanes

- **Goal:** the development of supersonic aircrafts, quiet enough to be allowed to fly supersonically over populated areas.
- The pressure signature created by the aircraft must be such that, when reaching the ground is below tolerated thresholds.



Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012, 44:505 – 26.

Motivation II: detection of polution sources



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Research article

Location and release time identification of pollution point source in river networks based on the Backward Probability Method

Alireza Ghane, Mehdi Mazaheri, Jamal Mohammad Vali Samani

Problem: Given an observation of the dynamical system at some time $T > 0$, to construct all the initial conditions which are compatible with it.

It is as “simple" as solving the equation

$$u_t + A(u) = 0$$

backwards in time:

$$u(T) \to u(0)$$

Note that the well-posedness of a Cauchy problem in the sense of Hadamard can be lost in the reverse sense of time

Wave equation: An example of hyperbolic time-reversibility

Solutions of smooth hyperbolic PDEs can be tracked back and forth following characteristic lines. This is the case for the classical wave or D'Alembert equation for elastic strings and membranes and acoustic waves

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \end{cases}$$

The equation is invariant by time-inversion

$$t \to -t.$$

The semigroup it generates it is actually a group of isometries.
The conservation of the energy can be red in both senses of time:

$$E(0) = E(T),$$

or, equivalently,

$$E(T) = E(0).$$

The equation, being well-posed in the backward sense, the inverse design problem has a unique solution, living in the same space as the target.

Consider:

$$\begin{cases} y_t - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \end{cases}$$

Highly dissipative effect:

$$\|y(T)\|^2_{L^2(\Omega)} = \sum_k e^{-2\lambda_k T} |\hat{y}_k^0|^2$$

This can also be re-written in terms of the energy of the initial data (at $t = 0$) recovered out of the final value at $t = T$:

$$\hat{y}_k^0 = e^{\lambda_k T} \hat{y}_k(T).$$

The same occurs for the solution of the Cauchy problem in the whole space:

$$y(x,t) = [G(\cdot,t) * y^0(\cdot)](x); \, G(x,t) = (4\pi t)^{-N/2} \exp(-|x|^2/4t).$$

A strongly irreversible process too.

The energy identity yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega y^2 dx + \int_\Omega |\nabla y|^2\, dx = 0, \tag{1}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega y^2 dx + \Lambda(t) \parallel y(t) \parallel^2_{L^2(\Omega)} = 0, \tag{2}$$

$$\Lambda(t) = \parallel \nabla y(t) \parallel^2_{L^2(\Omega)} / \parallel y(t) \parallel^2_{L^2(\Omega)} . \tag{3}$$

The frequency number $\Lambda(t)$ decreases along time

$$\Lambda(t) \leq \Lambda(0) = \Lambda_0, \ \forall t \geq 0.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega y^2 dx + \Lambda_0 \parallel y(t) \parallel^2_{L^2(\Omega)} \geq 0. \tag{4}$$

Therefore

$$||y(0)||^2 \leq \exp(2\Lambda_0 t)||y(t)||^2, \ \Lambda_0 = \parallel \nabla y^0 \parallel^2_{L^2(\Omega)} / \parallel y^0 \parallel^2_{L^2(\Omega)} .$$

This is estimate is sharp, the energy version of the Fourier series representation.

This is an interesting estimate that can be extended to a much broader class of diffusion processes to measure the complexity of the mixing of energy along different frequencies.

In this talk, we consider the inverse-time design problem for

- **Scalar conservation laws:**

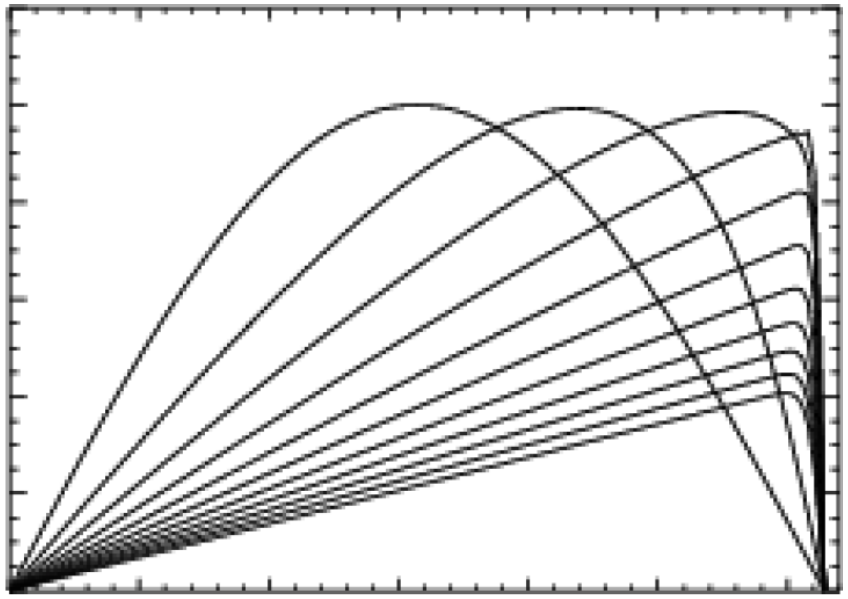
$$\partial_t u + \partial_x (f(u)) = 0 \qquad \text{in} \quad [0,T] \times \mathbb{R}$$

- **Hamilton–Jacobi equations:**

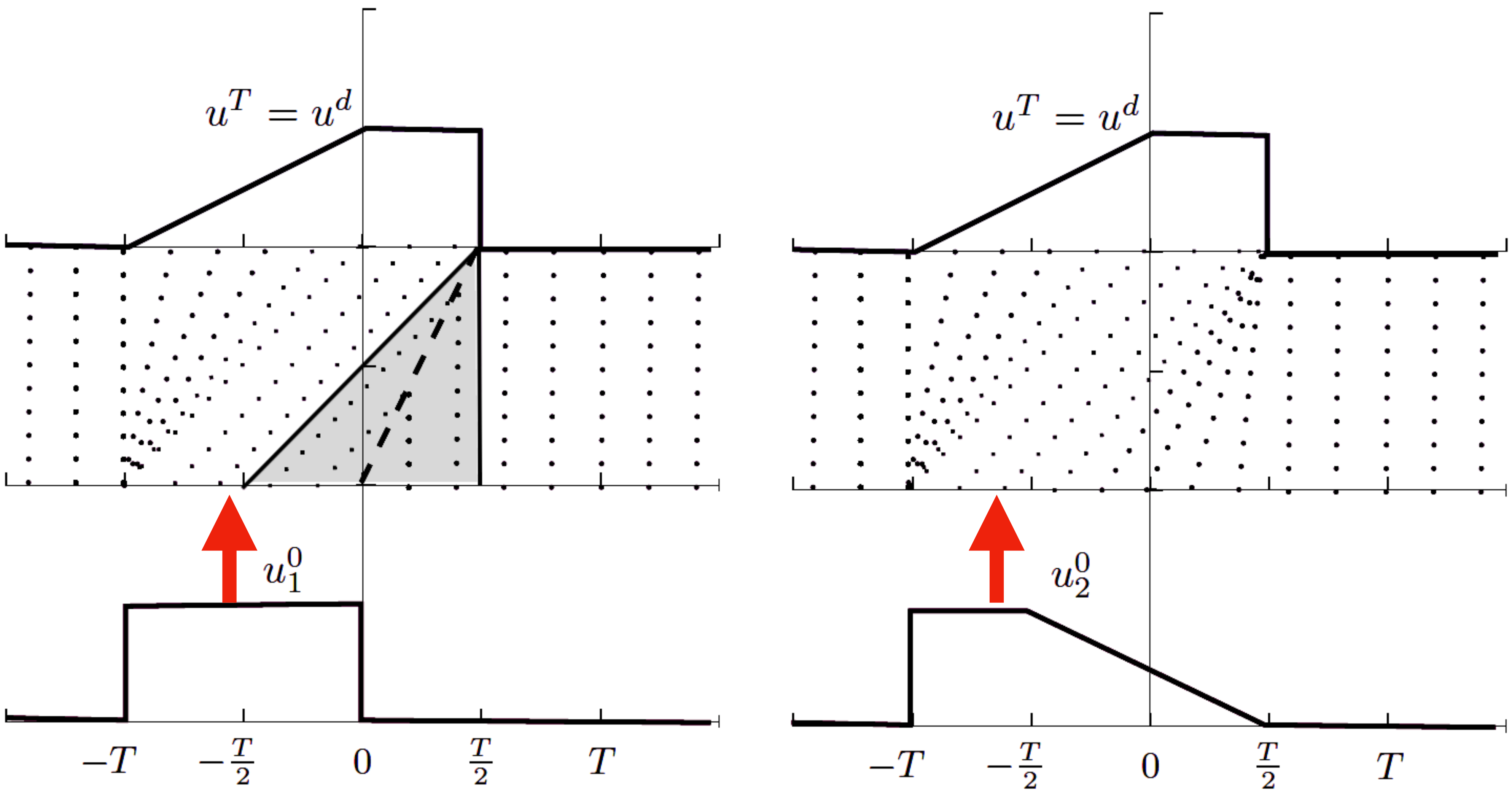
$$\partial_t u + H(\nabla_x u) = 0 \qquad \text{in} \quad [0,T] \times \mathbb{R}^N$$

Main difficulty:

Irreversibility occurs because of shock formation $\longrightarrow$ Lack of backward uniqueness.



Consider the one-dimensional Burgers equation $\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0$.



The entropy solutions with initial conditions $u_1^0 \neq u_2^0$ coincide at time $t = T$.

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) \in BV(\mathbb{R}) & \end{cases} \tag{1D-B}$$

For any initial condition $u_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$ **there exists a unique entropy solution** $u \in L^\infty([0,T] \times \mathbb{R}) \cap L^\infty([0,T]; BV(\mathbb{R}))$.

The entropy solution $u(t,x)$ can be obtained as the **zero viscosity-limit** of the nonlinear parabolic problem

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon \partial_{xx} u_\varepsilon + \partial_x \left(\frac{u_\varepsilon^2}{2} \right) = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ u_\varepsilon(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases}$$

This defines the forward entropy semigroup

$$\begin{array}{rccc} S_T^+ : & BV(\mathbb{R}) \cap L^\infty(\mathbb{R}) & \longrightarrow & BV(\mathbb{R}) \cap L^\infty(\mathbb{R}) \\ & u_0 & \longmapsto & S_T^+ u_0 := u(T, \cdot) \end{array}$$

which associates to any initial condition u_0, the entropy solution at time $t = T$.

Inverse design as an Optimal Control Problem

For any target $u_T \in L^\infty(\mathbb{R})$, we aim to solve the following optimal control problem

$$\min_{u_0} J_0(u_0) := \frac{1}{2}\int_{\mathbb{R}} (S_T^+ u_0(x) - u_T(x))^2 = \frac{1}{2}\|S_T^+ u_0 - u_T\|^2_{L^2(\mathbb{R})} \qquad \text{(Opt-Pb)}$$

Objectives:

- Characterize the class of **reachable targets**.
- Construction of a minimizer via a **backward-forward method.**
- Construction of **all the minimizers**.

It can be viewed as a **shooting problem**: Find u_0 such that $u(T) \sim u_T$.

Classical optimization methods, based on the use of gradient descent fail. The functional lacks differentiability when solutions develop shocks. The sensitivity of the functional dependent also on the sensitivity of shocks: shift differentiability [a] Descent algorithms need to be carefully tuned to lead to time inversion. [b]

[a] Ph. LeFloch & X. P. Xin, CPAM, 1993; A. Bressan & A. Marson, CPDE, 1995; F. Bouchut & F. James, NA, 1998; S. Ulbrich, SICON, 2002.

[b] N. Allahverdi, A. Pozo & E. Zuazua, ESAIM:COCV, 20216; L. Gosse & E. Zuazua, INdAM, 2017.

u_T is **reachable** at time T if there exists u_0 such that $S_T^+ u_0 = u_T$.

Characterization of reachable targets *[Gosse-Z, 2017], [Colombo-Perrollaz, 2019]*

A target u_T is reachable if and only of the following One-Sided Lipschitz Condition holds:

$$\partial_x u(x,t) \leq \frac{1}{T} \qquad \forall x_1, x_2 \in \mathbb{R} \tag{OSLC}$$

It is also known as the Oleinik Condition.

Splitting of the phase-space:

$$BV(\mathbb{R}) = \{\text{Reachable targets for } T > 0\} \cup \{\text{Unreachable targets for } T > 0\}.$$

Note that the class of reachable states shrinks as T grows enhancing time-irreversibility.

A natural way of recovering a candidate for initial datum is to solve the equation backwards in time.

We define the backward entropic semigroup $S_t^- : BV(\mathbb{R}) \to L^\infty(\mathbb{R})$:

$$\begin{cases} \partial_t v_\varepsilon + \varepsilon \partial_{xx} v_\varepsilon + \partial_x \left(\dfrac{v_\varepsilon^2}{2} \right) = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ v_\varepsilon(T,x) = u^T(x) & x \in \mathbb{R}. \end{cases}$$

In fact

$$S_t^- u^T(x) := S_t^+ \tilde{u}^T(-x)$$

where S_t^+ is the forward entropy operator and

$$\tilde{u}^T(x) = u^T(-x).$$

$$\min_{u_0 \in BV(\mathbb{R})} J_0(u_0) := \frac{1}{2} \int_{\mathbb{R}} (u_T(x) - S_T^+ u_0(x))^2 \qquad \text{(Opt-Pb)}$$

Theorem (Liard-Z, 2019)

* $u_0^* = S_T^- u^T$ is a minimizer obtained by backward entropic resolution.

* It either leads the final target u_T when it is reachable or otherwise into its L^2-projection.

* The optimal control problem admits multiple optimal solutions: the initial datum $u_0 \in BV(\mathbb{R})$ is a minimizer of (Opt-Pb) if and only if it satisfies

$$S_T^+ u_0 = S_T^+ (S_T^- u^T).$$

Remark:

- A full-characterization of such initial conditions is given in [Colombo-Perrolaz, 2019].
- This is better understood in the context of Hamilton-Jacobi equations.

The Optimal inverse design

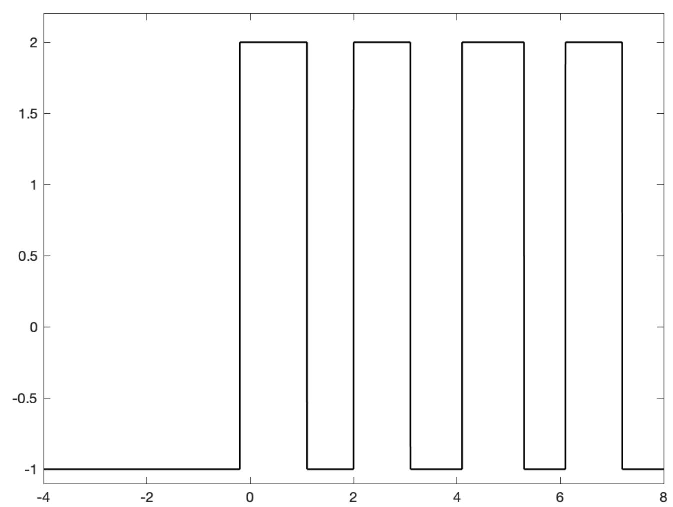


Figure: An unreachable target $u_T \in BV(\mathbb{R})$

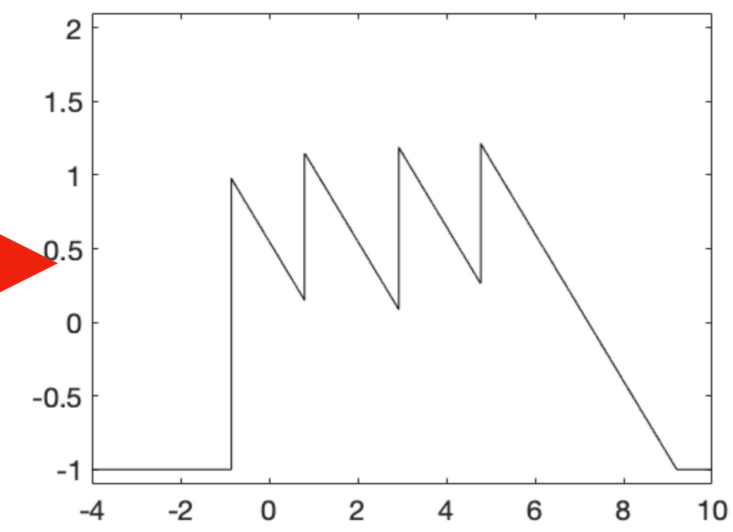


Figure: The backward entropy initial condition $u_0^* = S_T^- u^T$

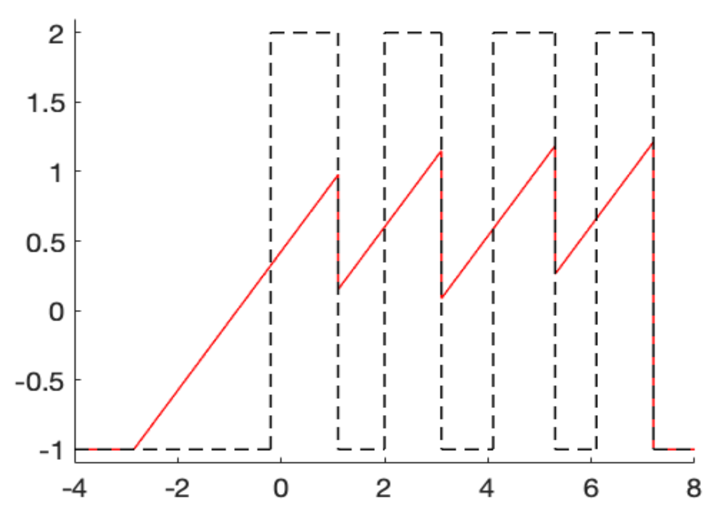


Figure: The closest reachable target to u^T for the L^2 distance, i.e. $\tilde{u}^T = S_T^+ u_0^*$

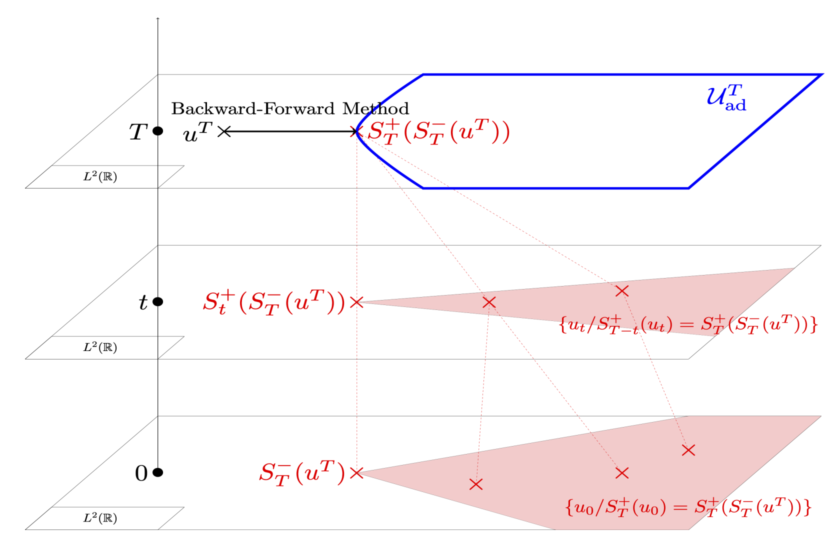
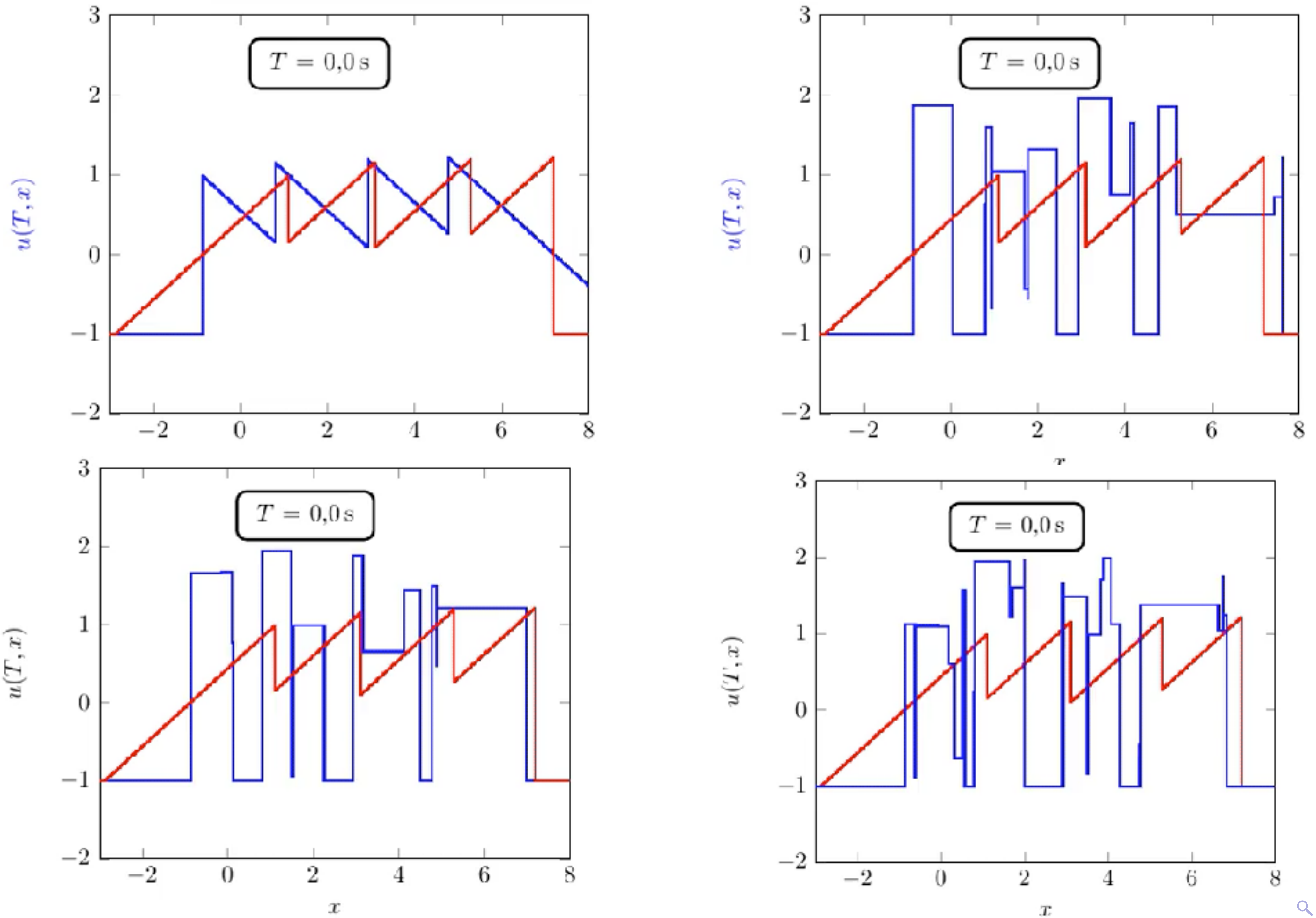
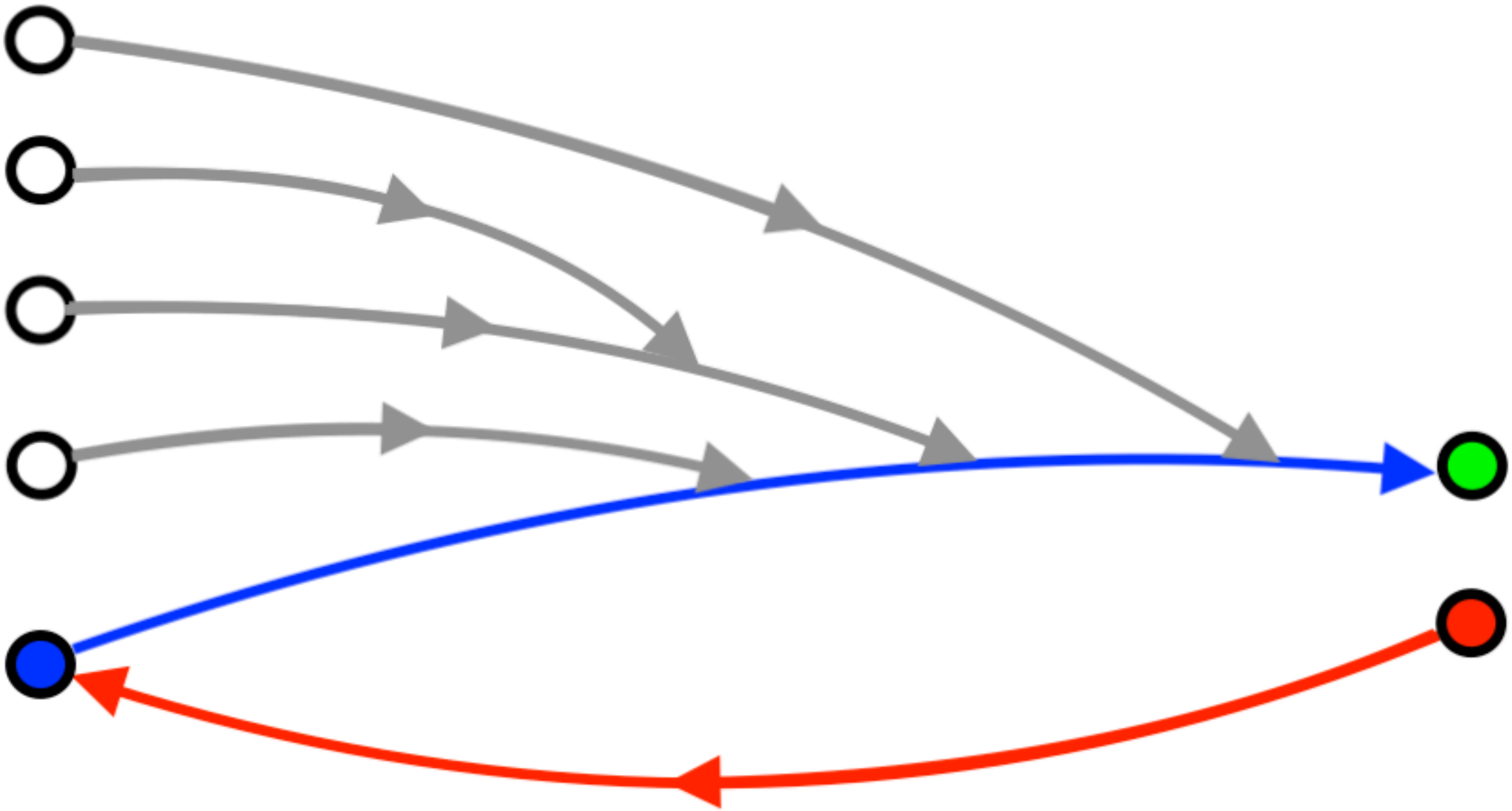


Figure: The minimizer u_0 might not be unique.

Different Optimal inverse designs





Backward-Forward resolution and Least Squares lead to the same inversion

In $1-d$ a scalar conservation law of the form

$$\partial_t v + \partial_x[f(v)] = 0 \tag{SCL}$$

can be transformed, by integration, in a Hamilton–Jacobi equation

$$\partial_t u + f(\partial_x u) = 0 \tag{HJ}$$

$v(t,x)$ solves the scalar conservation law iff

$$u(t,x) = \int_{-\infty}^{x} v(t,y)dy$$

solves the Hamilton-Jacobi equation.

Time-irreversibility

Solutions to the Hamilton-Jacobi equation (HJ) also **develop singularities**, which results in the ill-posedness of the backward Cauchy problem and the lack of backward uniqueness.

P. M. Cannarsa, W. Chen, 2021
F. Ancona, P. M. Cannarsa, K. T. Nguyen, 2016

$$\text{minimize } J_{t,x}[\alpha(\cdot)] := \int_{T-t}^{T} L(x(s),\alpha(s))dt + \phi(x(T))$$

over the admissible controls $\alpha(\cdot) \in L^1((0,T);\mathbb{R}^m)$, subject to

$$\begin{cases} \dot{x}(s) = f(x(s),\alpha(s)) & s \in [T-t,\ T] \\ x(T-t) = x \end{cases}$$

The **value function**

$$u(t,x) := \inf_{\alpha(\cdot)} J_{t,x}[\alpha(\cdot)]$$

is the viscosity solution to a **Hamilton–Jacobi** equation

$$\begin{cases} \partial_t u + H(x,\nabla_x u) = 0 & (t,x) \in [0,T] \times \mathbb{R}^N \\ u(0,x) = \phi(x) & x \in \mathbb{R}^N \end{cases}$$

$$H(x,p) = \max_{q\in\mathbb{R}^m} \{f(x,q)\cdot p - L(x,q)\} \quad \text{for any } p \in \mathbb{R}^N \text{(Hamiltonian).}$$

Identifying the initial condition ϕ in the HJ equation $\sim$ identifying the final cost in the minimization problem.

Inverse-time design for Hamilton–Jacobi equations

C. Esteve and E. Zuazua. *The inverse problem for Hamilton–Jacobi equations and semiconcave envelopes*, *SIAM J. Math. Anal.* , 2020.

$$\partial_t u + H(\nabla_x u) = 0 \quad \text{in } [0, T] \times \mathbb{R}^N; \; u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^N.$$

Objectives

- **Characterize reachable targets**
- **Approximate non-reachable targets.**
- **Initial-condition reconstruction.**
- **Interpretation in terms of optimal control.**

Background:

- The **Hamiltonian** $H : \mathbb{R}^N \to \mathbb{R}$ is usually considered to be convex (e.g. positive definite quadratic forms).
- The **initial datum** $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a given Lipschitz function.
- The vanishing visicosity method leads to viscosity solutions letting $\varepsilon \to 0^+$ in

$$\partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + H(\nabla_x u_\varepsilon) = 0, \qquad u_\varepsilon(0, x) = u_0(x)$$

Crandall-P.L.Lions, 1980's: Vanishing viscosity method $\to$ Existence, Uniqueness and Stability.

$$\begin{cases} \partial_t u + H(\nabla_x u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{HJ}$$

The unique viscosity solution is given by the **Hopf-Lax formula**:

$$u(t,x) = \min_{z \in \mathbb{R}^N} \left[u_0(z) + t\, H^* \left(\frac{x-z}{t} \right) \right] \qquad \text{for all } (t,x) \in (0,T] \times \mathbb{R}^N$$

where function H^* is the Legendre-Fenchel transform of H, i.e.

$$H^*(q) = \max_{p \in \mathbb{R}^N} [q \cdot p - H(p)], \qquad \text{for } q \in \mathbb{R}^N.$$

The Hopf-Lax formula links the Hamilton–Jacobi equation with an associated Optimal Control Problem

The unique viscosity solution $u(t,x)$ is the value function of the Optimal Control Problem

$$\underset{\alpha(\cdot)}{\text{minimize}} \int_{T-t}^{T} H^*(\alpha(s))ds + u_0(x(T)) \quad \text{s.t.} \quad \begin{cases} \dot{x}(s) = \alpha(s) \\ x(T-t) = x \end{cases}$$

The same as in 1-d scalar conservation laws we can define both the viscosity forward and backwards dynamics by the vanishing viscosity method:

Forward viscosity dynamics:

$$\begin{array}{rccc} S_T^+ : & \mathrm{Lip}(\mathbb{R}^N) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^N) \\ & u_0 & \longmapsto & S_T^+ u_0 := u(T, \cdot) \end{array}$$

where $u(T, \cdot)$ is the unique viscosity solution to HJ at time $t = T$ with initial condition.

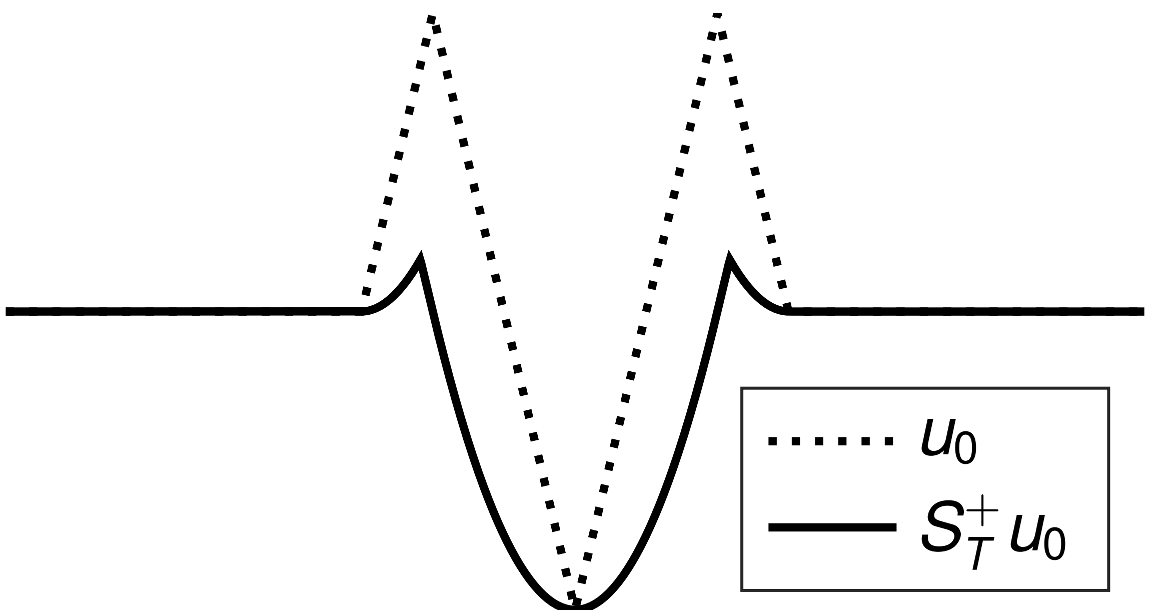
Backward viscosity dynamics:

$$\begin{array}{rccc} S_T^- : & \mathrm{Lip}(\mathbb{R}^N) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^N) \\ & u_T & \longmapsto & S_T^- u_T := u(0, \cdot) \end{array}$$

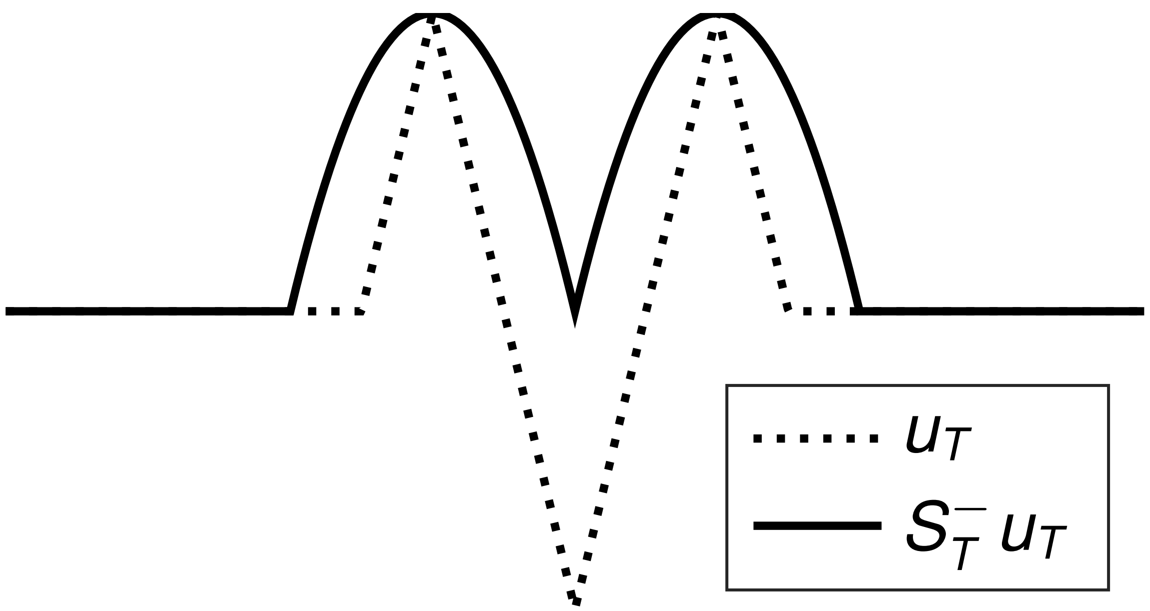
where $u(0, \cdot)$ is the unique backward viscosity solution to HJ at time $t = 0$ with final condition.

Property

1. For any $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$ and any $T > 0$, the function $S_T^+(x)$ is **semiconcave**: $\exists C > 0$ such that $D^2 f(x) \leq C$, $\forall x \in \mathbb{R}^N$..
2. For any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ and any $T > 0$, the function $S_T^-(x)$ is **semiconvex**: $\exists C > 0$ such that $D^2 f(x) \geq -C$, $\forall x \in \mathbb{R}^N$.



semiconcave

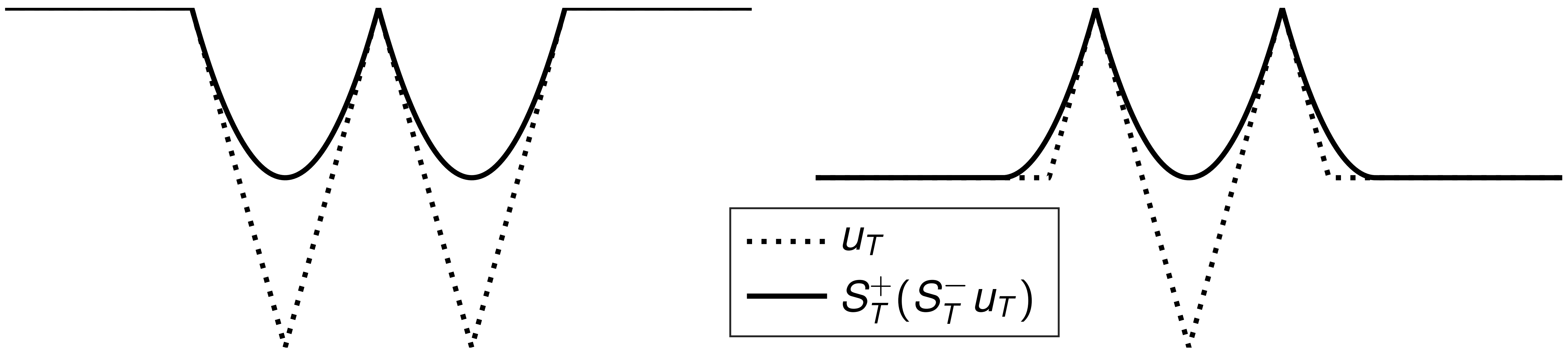


semiconvexe

Theorem

Let $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ and $T > 0$. Then u_T is reachable if and only if

$$S_T^+(S_T^- u_T) = u_T.$$



Two unreachable targets

- In order to be reachable, the target u_T needs to be semiconcave.
- The sharp semiconcavity condition for reachability depends on the Hamiltonian H and the time-horizon T.

Backward-forward criterion

Idea of the proof

The backward-forward reachability criterion is a direct consequence of the following property:

For any $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ and $T > 0$, we have

$$S_T^+ \circ S_T^- \circ S_T^+ \big(u_0(x)\big) = S_T^+ u_0(x).$$

$$\begin{tikzcd}
u_0 \arrow[drr, bend left, "S_T^+"] & & \\
S_T^-(S_T^+ u_0) = \tilde{u}_0 \arrow[rr, bend left, "S_T^+"] & & S_T^+ u_0 = u_T \arrow[ll, bend left, "S_T^-"]
\end{tikzcd}$$

What if the target u_T is **NOT** reachable?

Backward-Forward resolution $\neq$ Least Squares

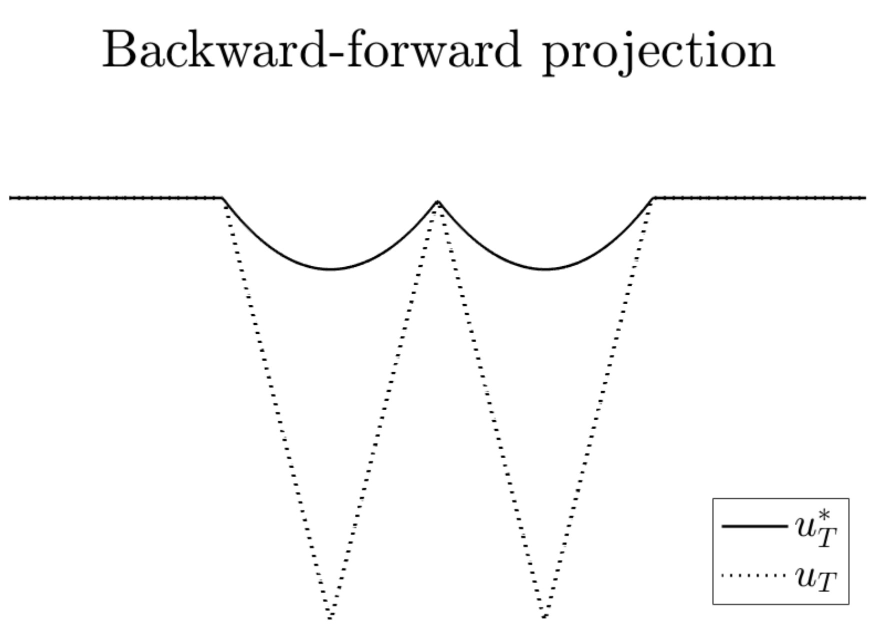
Different criteria lead to different inversions

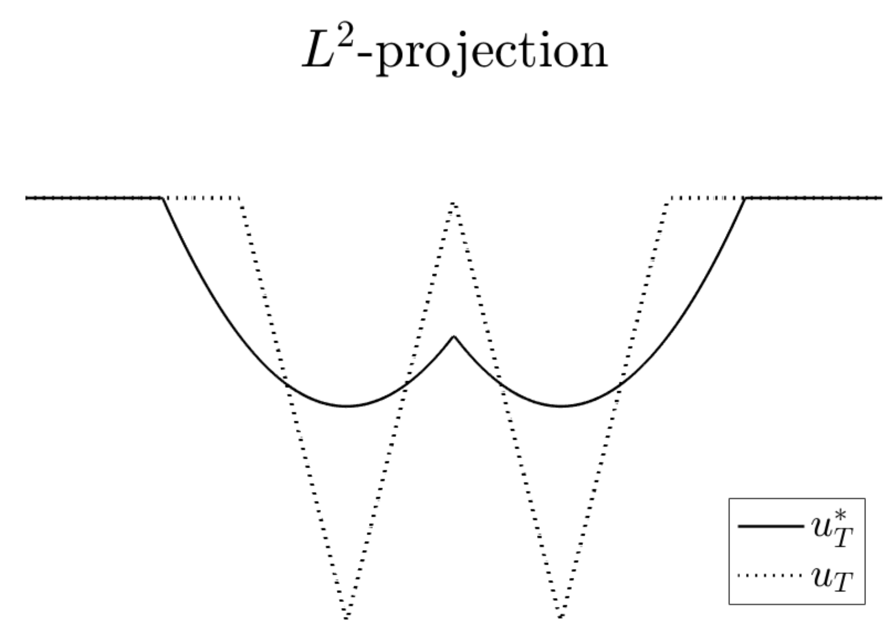
- **Backward-forward projection:**

$$u_T^* := S_T^+(S_T^- u_T)$$

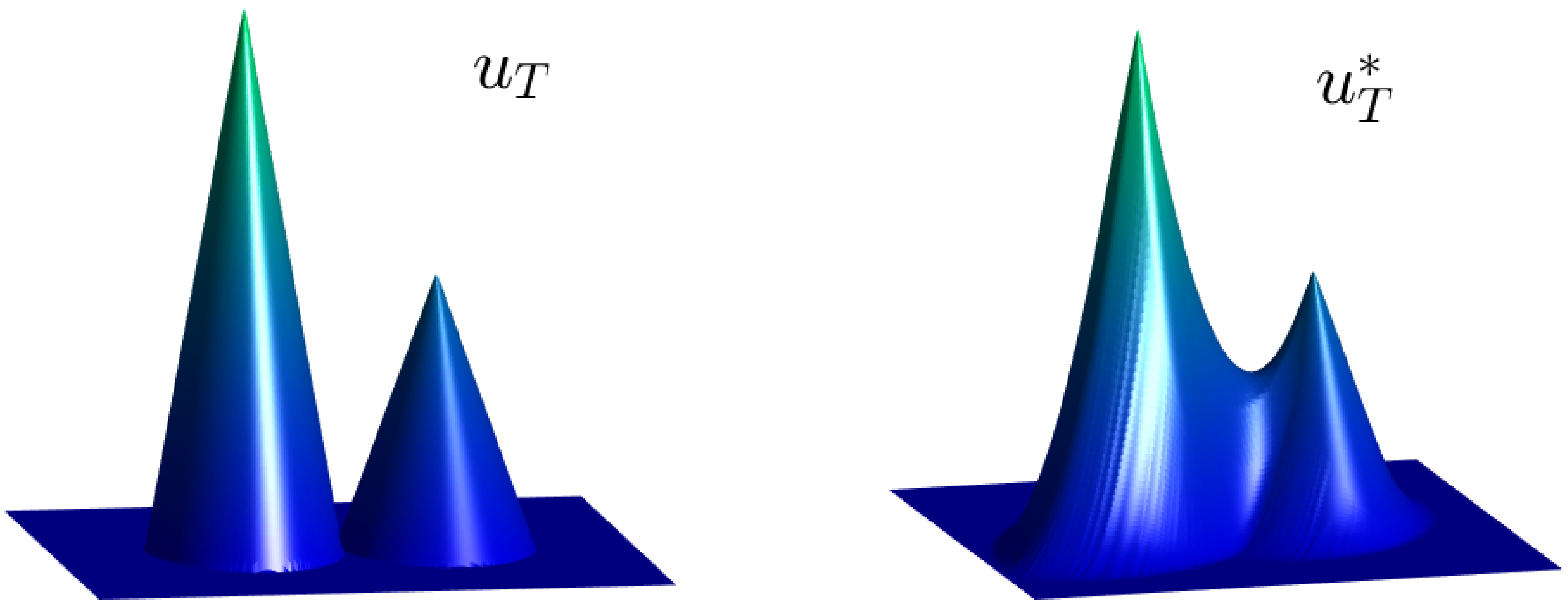
- L^2**-projection:** Consider the optimization problem

$$\underset{u_0 \in \mathrm{Lip}(\mathbb{R}^N)}{\text{minimize}} \; \mathcal{J}_T(u_0) := \|S_T^+ u_0 - u_T(\cdot)\|^2_{L^2(\mathbb{R}^N)}$$



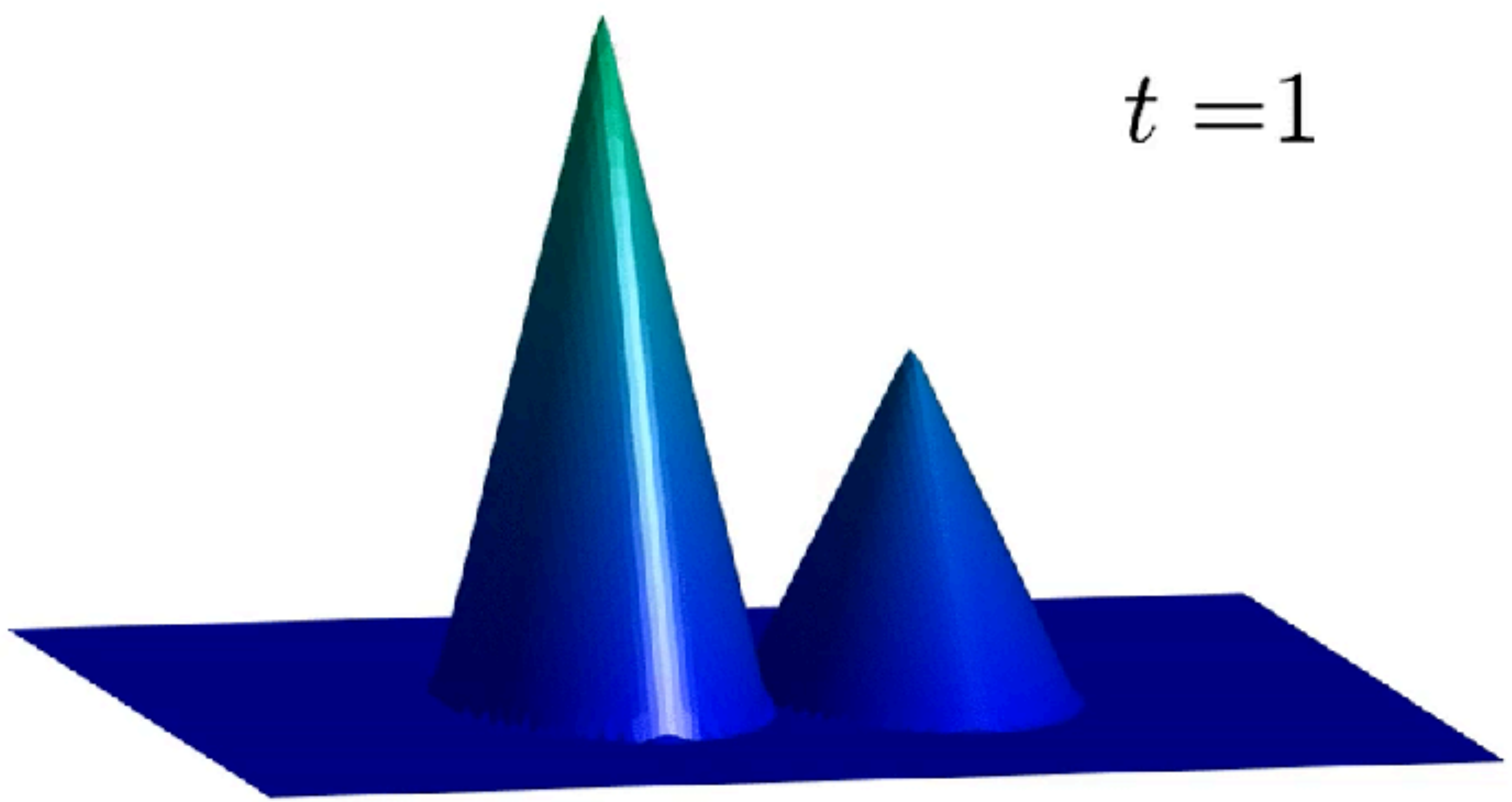


For any given target $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the function $u_T^* = S_T^+(S_T^- u_T)$ is the **smallest reachable target bounded from below by u_T.**



Left: unreachable target u_T. Right: Its A^{-1}/T–semiconcave envelope u_T^*.

Geometric properties of u_T^* (semiconcave envelopes)



Theorem (Esteve-Z, 2020)

Let

$$H(p) = \frac{\langle Ap, p\rangle}{2} \quad \text{for some matrix } A > 0.$$

Then, for any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the function $u_T^* = S_T^+(S_T^- u_T)$ is the viscosity solution to the obstacle problem

$$\min\left\{v - u_T,\ -\lambda_N\left[D^2 v - \frac{A^{-1}}{T}\right]\right\} = 0. \tag{5}$$

- Here, $D^2 v$ denotes the Hessian matrix of v; and for a symmetric matrix X, $\lambda_N[X]$ denotes its greatest eigenvalue.
- Observe that for T large, equation (5) is an approximation of the equation for the **concave envelope** of u_T

$$\min\left\{v - u_T,\ -\lambda_N\left[D^2 v\right]\right\} = 0.$$

- For any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, we call $u_T^* := S_T^+(S_T^- u_T)$ the $\frac{A^{-1}}{T}$**–semiconcave envelope** of u_T in $\mathbb{R}^N$.

Theorem (Esteve-Zuazua, 2020)

Let $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ be a reachable target and set the function $\tilde{u}_0 := S_T^- u_T$. Then, for any $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$, the two following statements are equivalent:

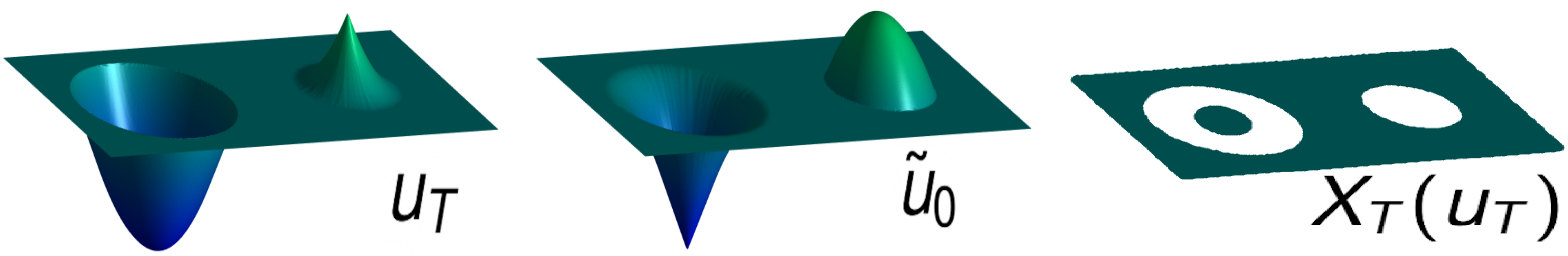
(i) $S_T^+(u_0) = u_T$;

(ii) $u_0(x) \geq \tilde{u}_0(x), \ \forall x \in \mathbb{R}^N$ and $u_0(x) = \tilde{u}_0(x), \ \forall x \in X_T(u_T)$,

where $X_T(u_T)$ is the subset of $\mathbb{R}^N$ given by

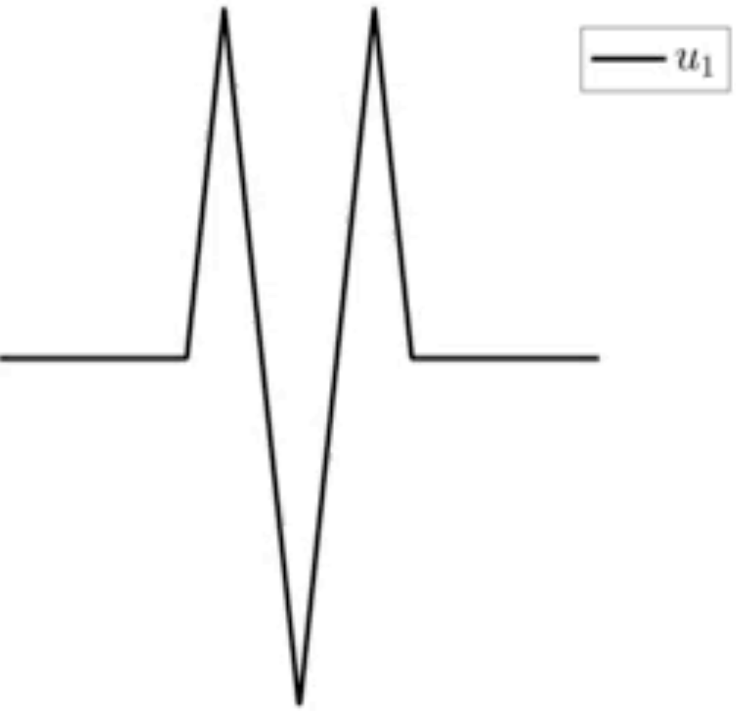
$$X_T(u_T) := \left\{ z - T\, H_p(\nabla u_T(z));\ \forall z \in \mathbb{R}^N \text{ such that } u_T(\cdot) \text{ is differentiable at } z \right\}.$$

Observe that the set of u_0's s. t. $S_T^+(u_0) = u_T$ is a convex cone with $\tilde{u}_0$ as vertex.

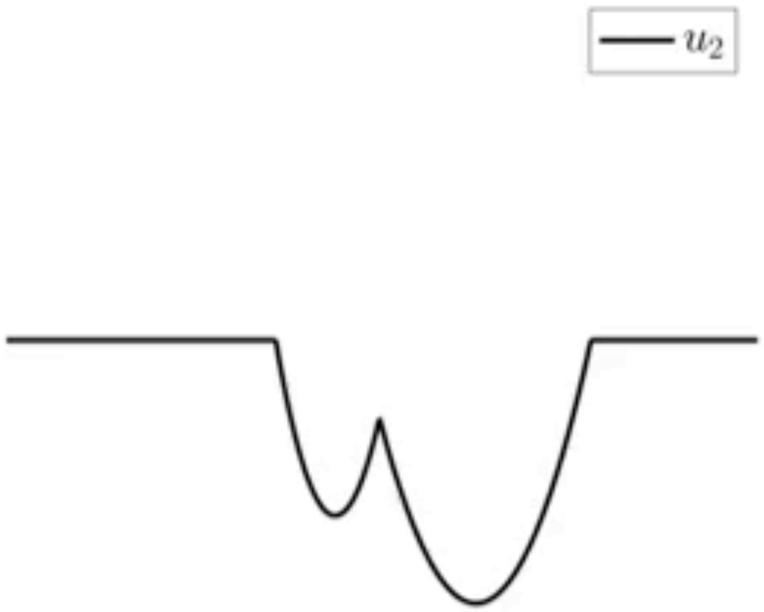


This leads to a simpler characterization of the class of initial data compatible with a final target for the 1-d Burgers equations (Colombo-Perrolaz)

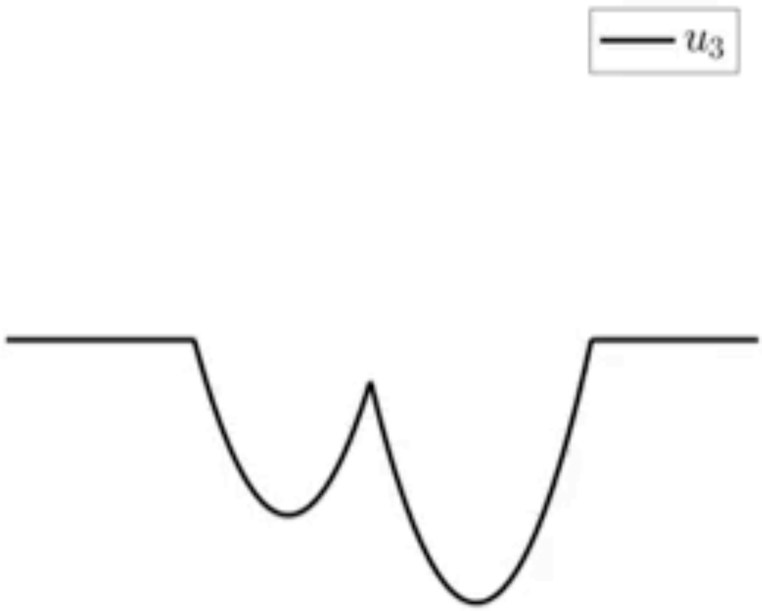
Backward-Forward



unreachable



unreachable



reachable

Conclusions, open problems and perspectives

- Same questions for all nonlinear evolution PDE.
- The least square approach $\equiv$ hard to implement numerically.
- The backward-forward methodology needs to be implemented with care, adapting the artificial viscosity to the sense of time.
- These two approaches do not necessarily to the same inversion.
- For more complex systems one may not employ OSLC or semiconcavity conditions that allow to characterize the reachable sets. T

Open problems:

1. Convex-concave fluxes f (pedestrian flows) / Non-convex Hamiltonians.
2. Multi-dimensional Conservation-Laws.
3. Systems of Conservation Laws.
4. Sharp semi-concavity conditions for general convex Hamiltonians.
5. Space-depending Hamiltonians.
6. More general systems: mixed hyperbolic-parabolic type...
7. Applications to Reinforcement Learning.

Note that backward-forward resolution is always a fast and good first hint for many problems that would be hard (or impossible) to address otherwise

Question: What if the target is NOT reachable?

We aim to construct a reachable function which is **as close as possible** to the given target.

We study two different projections:

- **The backward-forward projection:** Esteve-Z, SIMA 2020

$$\begin{tikzcd}
S_T^- u_T \arrow[rr, bend left, "S_T^+"] & & u_T^* = S_T^+(S_T^- u_T) \\
& & u_T \arrow[ull, bend left, "S_T^-"]
\end{tikzcd}$$

- **The L^2-projection:** *work in progress.*

$$u_T^* = S_T^+ u_0^*, \quad \text{where } u_0^* \text{ is the solution to the optimization problem}$$

$$\underset{u_0 \in \mathrm{Lip}(\mathbb{R}^N)}{\text{minimize}} \ \mathcal{J}_T(u_0) := \| S_T^+ u_0 - u_T(\cdot) \|^2_{L^2(\mathbb{R}^N)}.$$

The L^2-projection

Optimal inverse design

$$\underset{u_0 \in \mathrm{Lip}(\mathbb{R}^N)}{\text{minimize}} \ \mathcal{J}_T(u_0) := \|S_T^+ u_0 - u_T(\cdot)\|^2_{L^2(\mathbb{R}^N)}. \qquad \text{(OCP)}$$

This optimization problem can be cast as an optimal control problem in which the dynamics are given by (HJ), and the control is the initial condition.

- Existence of a minimizer for (OCP) can be proved by using compactness arguments, combined with the regularizing effect of the Hamilton-Jacobi equation (semiconcavity estimates).
- Uniqueness of the minimizer does NOT hold due to the lack of backward uniqueness of (HJ).

Gradient based methods

A solution to (OCP) can be approximated by means of a gradient descent algorithm:

$$u_0^{(n+1)}(x) = u_0^{(n)}(x) - \alpha_n \tilde{\Phi}_n(x),$$

where $\alpha_n > 0$, and

$$\tilde{\Phi}_n \approx D\mathcal{J}_T\left(u_0^{(n)}\right)$$

is a Lipschitz approximation of the gradient of the functional $\mathcal{J}_T$ at u_0.

The gradient of $\mathcal{J}_T$

For any $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$, the gradient of $\mathcal{J}_T$ at u_0 is given by the functional

$$w \in \mathrm{Lip}(\mathbb{R}^N) \longmapsto \partial_w \mathcal{J}_T(u_0) = 2\int_{\mathbb{R}^N} \left(S_T^+ u_0(x) - u_T(x)\right) \partial_w S_T^+ u_0(x) dx,$$

where $\partial_w S_T^+ u_0(x)$ is the directional Gateaux derivative of the operator S_T^+ with respect to u_0.

In a work in progress, we prove that $\partial_w S_T^+ u_0(x)$ exists and can be explicitly computed by means of the Hamiltonian system.

The gradient of the functional $\mathcal{J}_T$ at $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$ can also be given as

$$D\mathcal{J}_T(u_0) = \pi(0),$$

where $\pi(0)$ is the solution at time 0 of the backward conservative transport equation

$$\begin{cases} \partial_t \pi + \mathrm{div}(a(t,x)\pi) = 0 & (0,T) \times \mathbb{R}^N \\ \pi(T) = S_T^+ u_0 - u_T & \mathbb{R}^N \end{cases} \tag{6}$$

where

$$a(t,x) = H_p(x, \nabla u(t,x))$$

The solution to (6) exists and is unique, however, due to the low regularity of $a(t,x)$, the gradient $D\mathcal{J}_T(u_0) = \pi(0)$ is just a Radon measure in $\mathbb{R}^N$.

Remark:

Note that the equation (6) is the dual equation to the linear transport equation

$$\begin{cases} \partial_t v + a(t,x)\nabla_x v = 0 & (0,T) \times \mathbb{R}^N \\ v(0) = w \in \mathrm{Lip}(\mathbb{R}^N) & \mathbb{R}^N \end{cases}$$

resulting from the linearization of the Hamilton-Jacobi equation (HJ).

L^2-projection vs backward-forward approximation

Question: Are L^2-projection and the backward-forward approximation the same?

- **One-dimensional Burgers equation:** the L^2-projection of any target $v_T \in BV(\mathbb{R})$ onto the set of reachable targets is unique and is given by

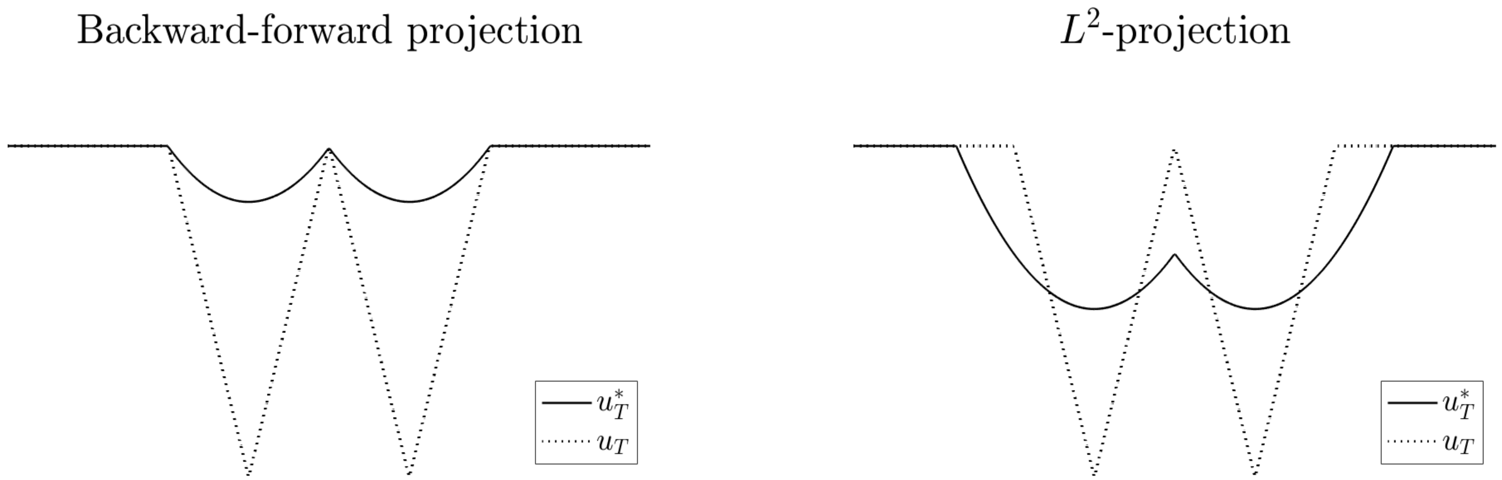
$$v_T^* = S_T^+(S_T^- v_T)$$

[see Liard-Z., 2021]

- **Hamilton-Jacobi equation:** the L^2-projection is different to the function after a backward-forward resolution of (HJ). The function

$$\tilde{u}_T := S_T^+(S_T^- u_T)$$

is actually the smallest reachable target bounded from below by u_T. *[see Esteve-Z, 2020]*



Reachability condition: non-smooth Hamiltonians

Let us consider the one-dimensional case in space and the Hamiltonian

$$H(p) = |p| \qquad \text{for } p \in \mathbb{R}.$$

- The reachability condition for a target u_T cannot be determined in terms of the semiconcavity of u_T since, in this case, the solution to (HJ) is no longer semiconcave[1]
- Nonetheless, for the case $H(p) = |p|$, we can characterize the reachable set differently as in the following theorem.

Theorem:

Let $u_T \in \mathrm{Lip}(\mathbb{R})$, $H(p) = |p|$ and $T > 0$. Then, the following statements are equivalent:

1. u_T is reachable;
2. For any $x \in \mathbb{R}$, if x is a local minimum of u_T, then there exists $x_0 \in \mathbb{R}$ such that $|x - x_0| \leq T$ and $u_T(x) \leq u_T(x)$ for all $y \in B(x_0, T)$.

(see an illustration in the following slide)

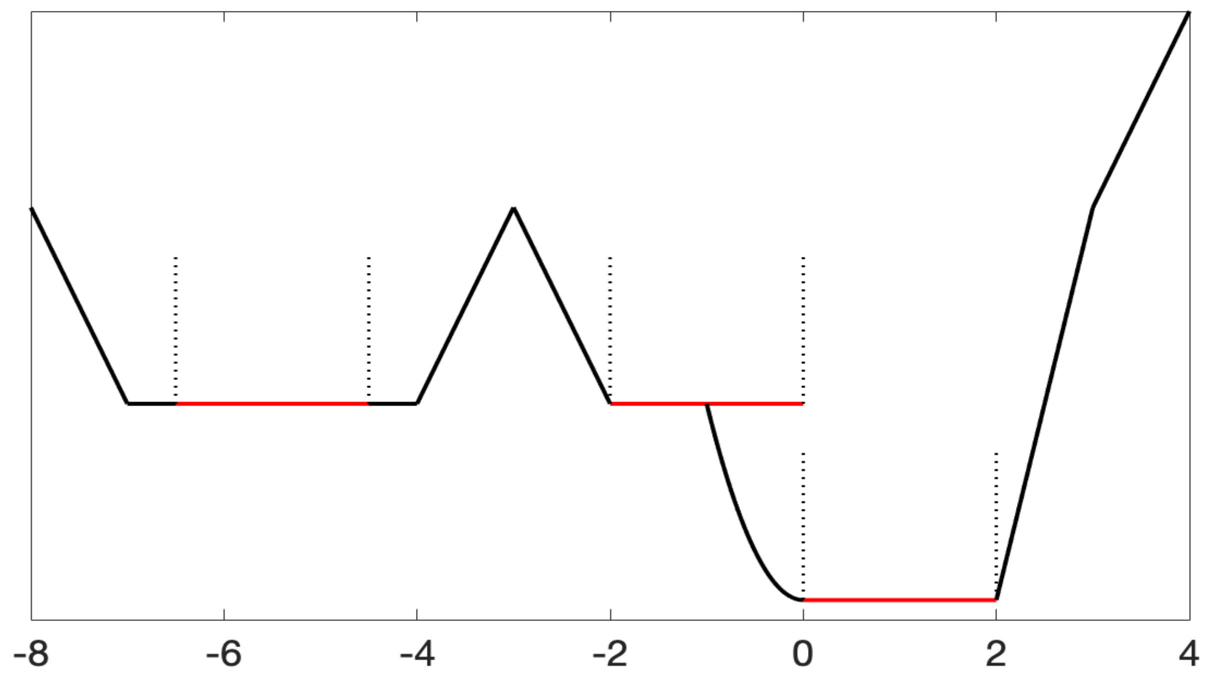
[1] Recall that the semiconcavity of the solution holds under the assumption $\partial_{pp} H(p) \geq c$, for some $c > 0$.

The reachability condition in the above Theorem can be proved by using the Hopf-Lax formula, which, for the case $H(p) = |p|$ reads as

$$S_T^+ u_0(x) = \min_{y \in B(x,T)} u_0(y) \quad \text{and} \quad S_T^- u_T(x) = \max_{y \in B(x,T)} u_T(y),$$

for the forward and backward viscosity operator respectively.

An example of reachable target for the Hamiltonian $H(p) = |p|$ and $T = 1$.



In view of the above theorem, we have that **concave functions** $u_T : \mathbb{R} \to \mathbb{R}$ **are reachable for all** T**.**