Li–Yau type inequality for curves and applications

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Organization

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1. Introduction: bending energy and self-intersection.
2. Main result: Li–Yau type multiplicity inequality.
3. Applications: elastic flow, elastic network, elastic knot.

Main reference:

- Miura: Li–Yau type inequalities for curves in any codimension, arXiv:2102.06597.

Introduction

1. Introduction: bending energy and self-intersection.

What is the bending energy?

Bending energy:

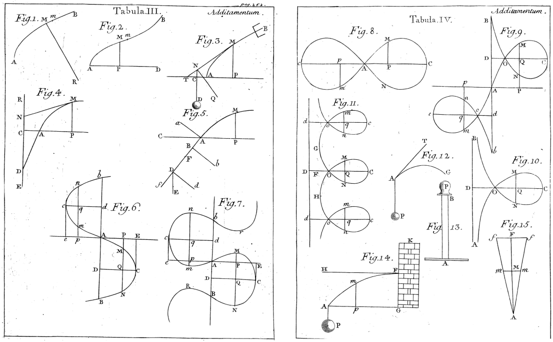
- A quantity measuring how a curve (or surface) bend,

$$\int_\gamma |\kappa|^2 ds,$$

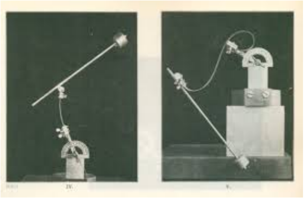
  where γ is a curve in $\mathbf{R}^n$, s arclength, $\kappa := \partial_s^2 \gamma$ curvature vector.
  - D. Bernoulli (1742), L. Euler (1744), ..., M. Born (1906), ...

- Minimize bending energy in a certain class of curves.
  $\rightsquigarrow$ The minimizer often realizes the shape of a real object.
- Critical point under fixed-length constraint is called (Euler's) elastica.

Example 1: Elastic rod



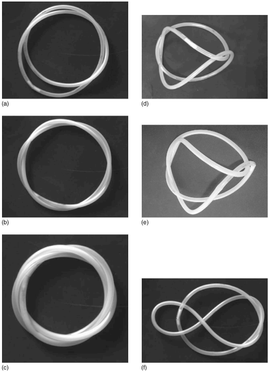
[Euler 1744]



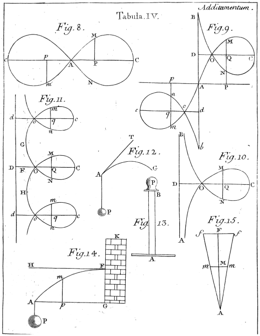
[Born 1906]

Example 2: Self-intersecting elastic curves

- The effect of self-intersection yields more complicated shapes.
- See experiments on elastic knots (and Euler's figures again).



[Gallotti–Pierre-Louis '07, Phys Rev E]



[Euler 1744]

- Here we mainly address a more fundamental problem, Li–Yau inequality.

Main result

2. Main result: Li–Yau type multiplicity inequality.

Classical Li–Yau inequality for surfaces

- Bending energy (Willmore energy) for closed surfaces $\Sigma^2 \subset \mathbf{R}^n$:

$$W[\Sigma] := \int_\Sigma |H|^2 \quad (H\text{: mean curvature vector}).$$

  - Note: W is scale invariant & minimized by a round sphere.
- Σ has a point $p \in \Sigma \subset \mathbf{R}^n$ of multiplicity k if Σ passes p at least k times.

Theorem (Li–Yau'82, Invent Math)

If a closed surface $\Sigma^2 \subset \mathbf{R}^{n\geq 3}$ has a point of multiplicity k, then $W[\Sigma] \geq 4\pi k$.

Remarks:

- Sharp: $W \approx 4\pi k$ for nearly k-times covered spheres.
- If $W[\Sigma] < 8\pi$, then Σ must be embedded.
- Fundamental tool. [Kuwert–Schätzle'14, Ann Math], [Marques–Neves'14, Ann Math]

- The proof heavily relies on 2D.
- How about 1D curves? $\rightsquigarrow$ new phenomena due to low dimensionality.

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Normalized bending energy

Goal: establish a Li–Yau type inequality for closed curves γ.

- BUT the bending energy $B[\gamma] = \int_\gamma |\kappa|^2 ds$ is not scale-invariant!
- Consider the scale-invariant normalized bending energy $\bar{B}$:

$$\bar{B}[\gamma] := L[\gamma] \int_\gamma |\kappa|^2 ds.$$

(Equivalent to assuming $L[\gamma] = 1$.)

- This is a right quantity since for any closed curve γ in $\mathbf{R}^{n\geq 2}$,

$$\bar{B}[\gamma] \geq 4\pi^2,$$

where equality holds if and only if γ is a round circle.

Precise goal: find an inequality for γ with multiplicity k of the form

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Bending energy and multiplicity

- Let $\varpi^* = 28.109...$ be an explicit constant defined by elliptic integrals.

Theorem (M.)

Let γ be a closed H^2-curve in $\mathbf{R}^{n\geq 2}$ with a point of multiplicity $k \geq 2$. Then

$$\bar{B}[\gamma] \geq \varpi^* k^2,$$

where equality holds iff $n \geq 3$ or k is even and γ is a k-leafed elastica. On the other hand, if $n = 2$ and k is odd, then $\exists \varepsilon_k > 0$ such that

$$\bar{B}[\gamma] \geq \varpi^* k^2 + \varepsilon_k.$$

- Covers general (n, k) and optimal in many cases*.
  - ↿ Variational proof, using classification of elasticae [Langer–Singer'84, JDG].
- Nonoptimality is caused by a hidden algebraic obstruction.
  - ↿ Proof by a transcendency result involving ${}_2F_1$ [André'96, Crelle].

*Müller–Rupp 2021: Optimal estimate for $(n, k) = (2, 2)$. Polden 1996, von der Mosel 1998, Wheeler 2013,...: Non optimal estimate $\bar{B} \geq Ck^2$.

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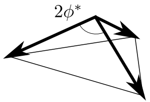
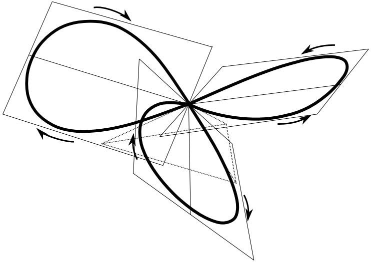
New optimal shapes are discovered

k**-Leafed elastica:** composed of k half-fold figure-eights of same length.

- 2-Leafed elastica $\Leftrightarrow$ (unique) figure-eight elastica.
- 3-Leafed elastica $\Leftrightarrow$ (unique) elastic propeller:





(This is why higher codimension $n \geq 3$ is needed for $k = 3$.)

- If $k \geq 4$, then k-leafed elasticae are not unique, but still strongly rigid.

Applications

3. Applications: elastic flow, elastic network, elastic knot.

Application 1: Elastic flow

Corollary (Embeddedness criterion)

For a closed curve γ in $\mathbf{R}^n$, if $\bar{B}[\gamma] < 4\varpi^*$, then γ is embedded.

Elastic flow: For $\lambda > 0$,

$$\partial_t \gamma = -2\nabla_s^2 \kappa - |\kappa|^2 \kappa + \lambda \kappa.$$

- $L^2(ds)$-gradient flow of $E_\lambda := B[\gamma] + \lambda L[\gamma] \rightsquigarrow$ energy decreases.
- 4th-order flow $\rightsquigarrow$ embeddedness may break in the middle.

Corollary (Optimal threshold for all-time embeddedness)

If $\frac{1}{4\lambda} E_\lambda[\gamma_0]^2 < 4\varpi^*$, then the elastic flow from γ_0 is embedded for all $t \geq 0$.

- Optimality: Figure-eight elastica is stationary and $\frac{1}{4\lambda} E_\lambda[\gamma]^2 = 4\varpi^*$.
- Easy but key fact: $\bar{B} \leq \frac{1}{4\lambda} E_\lambda^2$ in general. (Fixed-length flow more direct.)

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Application 2: Elastic network

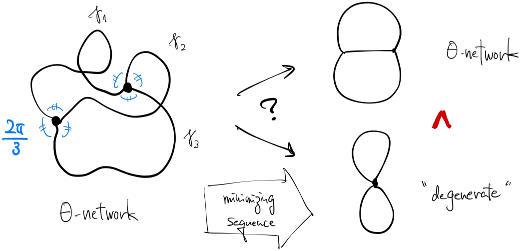
Theta-network: Triple of curves $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ meeting equal $\frac{2\pi}{3}$-angle.

$$E[\gamma] := \sum_{i=1}^{3}(B[\gamma_i] + L[\gamma_i]) = \sum_{i=1}^{3}\int_{\gamma_i}(|\kappa|^2 + 1)ds.$$

Theorem (Existence of minimal elastic Θ-network)

There exists a minimizer of E among all Θ-networks in $\mathbf{R}^n$.

- [Dall'Acqua–Novaga–Pluda '20, Indiana] proved $n = 2$, while $n \geq 3$ left open.



Application 3: Elastic knot

Theorem (Langer–Singer '84 JDG, '85 JLMS, '85 Topology)

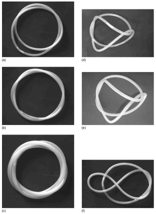
Any closed elastica in $\mathbf{R}^3$ is one of 1, 2, 3, or its multiple covering:

1. *Circle (planar).*
2. *Figure-eight elastica (planar).*
3. *Embedded torus knots (spatial, infinitely many).*

Among them, the only stable one is the one-fold circle.

How the elastic knots appear?

- Min B with suitable constraint on self-intersections:
  The idea goes back to [von der Mosel '98, Asymptot Anal]

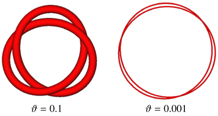


[Gallotti–Pierre-Louis '07, Phys Rev E]

Application 3: Elastic knot

Known results:

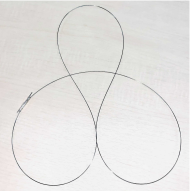
- The minimal elastic unknot is the one-fold circle.
- The minimal elastic trefoil is the two-fold circle.
  - Moreover true for minimal $(2, b)$-torus knots.
    [Gerlach–Reiter–von der Mosel '17, ARMA]



[Gerlach–Reiter–von der Mosel '17]

Conjectures on minimal elastic knots:

- Circular elastic knot conjecture.
  [Gallotti–Pierre-Louis '07]
- Teardrop-heart elastic knot conjecture.
  [Miura–Müller–Rupp, arXiv:2106.09549]



New conjecture on "stable" elastic knots:

- The second smallest stable elastic unknot is the elastic propeller.

The last two conjectures are strongly inspired by numerical studies in [Avvakumov–Sossinsky '14], [Bartels–Reiter '21].

Movie

Experiment: elastic unknots by a closed springy wire:

- Circle is stable (minimal).
- Figure-eight elastica is not stable in space (but stable in plane).
- Elastic propeller is stable!

Summary and future directions

Summary:

- Normalized bending energy $\bar{B}[\gamma] = L\int_\gamma |\kappa|^2 ds$ and multiplicity k:

$$\bar{B}[\gamma] \geq \varpi^* k^2.$$

- Optimal in many cases. Nonoptimal otherwise.
- $k=2$: Figure-eight elastica. $k=3$: Elastic propeller.

Open problems:

- Optimal shapes in remaining cases $n=2$ and odd $k \geq 3$?
- General $\int_\gamma |\kappa|^p$? General dim $M^m \subset \mathbf{R}^n$?
- Uniqueness of minimal elastic Θ-network? Shape?
- Proofs of the conjectures on elastic knots?

– Thank you.