Ricci Flow and pinched curvature on non-compact manifolds

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The Chinese University of Hong Kong

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The Asia-Pacific Analysis and PDE seminar

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Given a complete Riemannian manifold (M^n, g_0) with non-negative or positive **curvature**, *under what curvature condition we can classify M^n?*

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Suggesting: curvature should be relatively strong!

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Question: Any Gap Theorem of flat space under point-wise condition?

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2. Optimal since $\mathbb{CP}^n$ achieves the borderline case: $K \in [\frac{1}{4}, 1]$

Hamilton's Ricci flow

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4. Brendle-Schoen: Differentiable sphere Theorem under $1/4$-pinched

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Definition (Micallef-Moore)

The curvature Rm *is said to have non-negative isotropic curvature (*$\mathrm{Rm} \in \mathrm{C}_{PIC}$*) if the complexified sectional curvature* $\mathrm{K}^{\mathbb{C}}(\Sigma) \geq 0$ *for any totally isotropic 2-plane* $\Sigma \subset T_pM \otimes \mathbb{C}$ *(i.e.* $\langle v, v \rangle = 0$ *for* $v \in \Sigma$*).*

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Differentiable sphere Theorem implied by the following powerful Theorem:

Theorem (Brendle-Schoen, Brendle)

A closed manifold (M, g) *such that* $\mathrm{Rm} - \varepsilon I \in \mathrm{C}_{PIC1}$ *for some* $\varepsilon > 0$*, then the Ricci flow converges to spherical space-form after normalization.*

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Question: Is bounded curvature really necessary? Can we generalize further (analogous to sphere theorem)?

Hamilton pinching conjecture in 3D

Theorem (Chen-Zhu, Lott, Deruelle-Schulze-Simon, L.-Topping)

Suppose (M^3, g_0) *is complete manifold with* $\mathrm{Ric}(g_0) \geq \varepsilon_0 \mathrm{scal}(g_0)$ *for some* $\varepsilon_0 > 0$*, then* (M^3, g_0) *is compact or flat.*

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Question: What about higher dimension? What is the correct generalization of Ric in this spirit?

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Theorem (L.-Topping)

Suppose (M^n, g_0) is a complete non-compact manifold such that

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for some $\varepsilon_0 > 0$ and $\mathrm{Rm}(g_0) \in \mathrm{C}_{PIC2}$, then (M, g_0) is flat.

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4. rescaled limit solution $g_\infty(t)$ gives metric structure of tangent cone at infinity of (M, g_0) when $t \to 0$.

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- L.-Tam, Hochard, Simon-Topping, Lai, etc: non-collapsed with strong curv > -1

Remarks, contn

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2. General difficulty: basic maximum principle fails (due to failure of control at infinity)

Effective Existence theory under pinching

Theorem (L.-Topping)

Suppose (M^n, g_0) *is a complete non-compact manifold with* $n \geq 3$ *and*

$$\mathrm{Rm}(g_0) - \varepsilon_0 \mathrm{scal}(g_0) \cdot Id \in \mathrm{C}_{PIC1}$$

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for some $1 >> \varepsilon_0 > 0$*, then there exists a Ricci flow solution* $g(t)$ *on* $M \times [0, +\infty)$ *from* g_0 *such that*

1. $\mathrm{Rm}(g(t)) - \varepsilon_1 \mathrm{scal}(g(t)) \cdot Id \in \mathrm{C}_{PIC1}$;
2. $|\mathrm{Rm}(g(t))| \leq at^{-1}$ *for some* $a(n, \varepsilon_0) > 0$;
3. $\mathrm{scal}(g(t)) > 0$ *if* g_0 *is non-flat;*
4. $\mathrm{Rm}(g(t)) \in \mathrm{C}_{PIC2}$ *if* $\mathrm{Rm}(g_0) \in \mathrm{C}_{PIC2}$.

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$$g_k(t) = k^{-1} g(kt) \approx \text{Ricci flow from metric cone at infinity.}$$

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1. Short-time Existence theory non-rely on Shi's classical solution
2. Non-negative curvature preserved;
3. Pinched curvature is preserved: $\mathrm{Rm} - \varepsilon' \mathrm{scal} \cdot I \in \mathrm{C}_{PIC1}$ for all $t > 0$!
4. curvature becomes bounded instantaneously after it evolves (infinity propagation)
5. More importantly, estimates are scaling invariant: decay in c/t
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8. Argument makes rigorous using Brendle's Li-Yau-Hamilton's Harnack inequality.

Idea of construction

The existence is based on local construction:

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Theorem (Local existence of Ricci flow)

Suppose M is non-compact and $p \in M$ satisfies $B_{g_0}(p,2) \Subset M$ and

$$\mathrm{Rm}(g_0) - \varepsilon_0 \mathrm{scal}(g_0) \cdot I \in \mathrm{C}_{PIC1}$$

on $B_{g_0}(p,2)$, then there exists $T(n,\varepsilon_0), a(n,\varepsilon_0), \varepsilon_0'(n,\varepsilon_0) > 0$ and a Ricci flow solution $g(t)$ on $B_{g_0}(p,1) \times [0,T]$ so that

1. $|\mathrm{Rm}(g(t))| \leq a t^{-1}$;
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Moral Idea: the (almost) pinching will imply that $g(t)$ looks like sphere at large curvature region which contradicts with non-compactness

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Sub-question (crucial step in 3D): generalization of Schoen-Yau Theorem

Question

Suppose (M^n, g_0) *is complete non-compact manifold with*

$$\mathrm{Rm}(g_0) \in int\left(\mathrm{C}_{PIC1}\right),$$

then M *is diffeomorphic to* $\mathbb{R}^n$

THANK YOU!!