On an overdetermined problem involving the fractional Laplacian

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Asia-Pacific Analysis and PDE Seminar
May 2023

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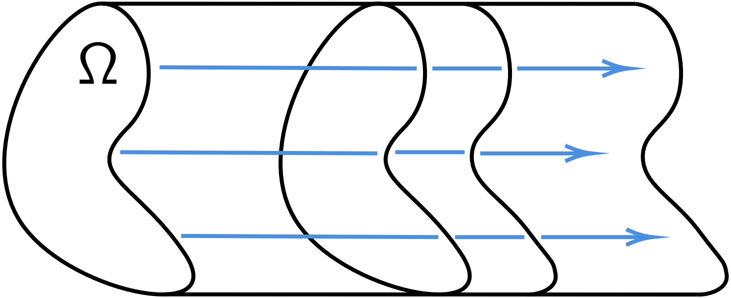

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$$\partial_\nu u = \text{ const.} \qquad \text{on } \partial\Omega.$$

This constant is a Lagrange multiplier corresponding to the assumption that u maximises the torsional rigidity:

$$\tau(\Omega) := \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\left(\int_\Omega v\,dx\right)^2}{\int_\Omega |\nabla v|^2\,dx}.$$

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Suppose that $\Omega \subset \mathbb{R}^n$ *is a bounded domain with* C^2 *boundary. If there exists a solution* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *that satisfies*

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then Ω is a ball.

Proof relies on the powerful technique now known as the *method of moving planes*.

The fractional Laplacian

- A nonlocal/integro-differential operator given by

$$(-\Delta)^s u(x) = c_{n,s}\mathrm{P.V.} \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) \frac{dy}{|y|^{n+2s}}$$

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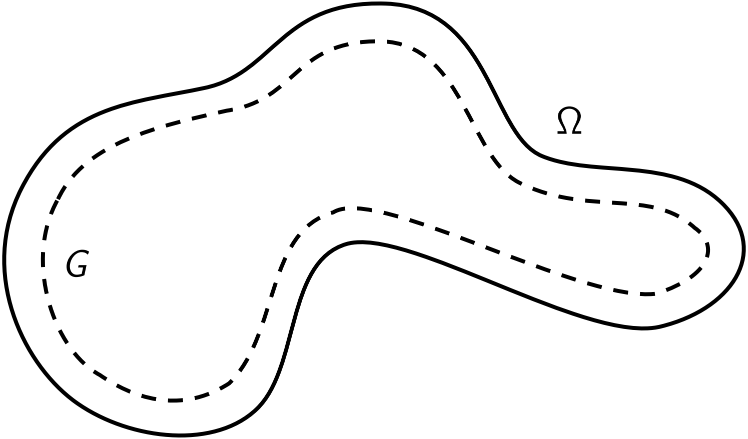
Parallel surface problem

Let $\Omega = G + B_R$ where

$$A + B := \{a + b \text{ such that } a \in A, b \in B\}.$$



Suppose that G is a bounded open set in $\mathbb{R}^n$ with C^1 boundary, $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, and $\Omega = G + B_R$.

Theorem (Dipierro, Poggesi, T, Valdinoci,'22)

Suppose that there exists a non-negative function $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^n)$ that is not identically zero and satisfies

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ u = const. & \text{on } \partial G. \end{cases}$$

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- $s = 1$: [Ciraolo, Magnanini, Sakaguchi, '15]
- $0 < s < 1$ and $f \equiv 1$: [Ciraolo, Dipierro, Poggesi, Pollastro, Valdinoci, '21]

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Let $e \in \mathbb{S}^{n-1}$, $\mu \in \mathbb{R}$, and $T_\mu = \{x \cdot e = \mu\}$. Define

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Goal: Prove that $v_\mu \equiv 0$ when $\mu = \lambda :=$ critical time.

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By linearity of $(-\Delta)^s$, for all $\mu \in (\lambda, \Lambda)$,

$$\begin{cases} (-\Delta)^s v_\mu + c_\mu v_\mu = 0, & \text{in } \Omega'_\mu \\ v_\mu \geqslant 0, & \text{in } H'_\mu \setminus \Omega'_\mu \end{cases}$$

where

$$c_\mu(x) = \begin{cases} -\frac{f(u(x)) - f(u(x'_\mu))}{u(x) - u(x'_\mu)}, & \text{if } u(x) \neq u(x'_\mu) \\ 0, & \text{if } u(x) = u(x'_\mu). \end{cases}$$

where Ω'_μ is the reflection of the RHS of Ω across T_μ, x'_μ is the reflection of x across T_μ, and H'_μ is the halfspace on the LHS of T_μ.

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In the local case, Step 1 follows immediately from the maximum principle. However, the maximum principle for nonlocal operators requires that $v_\mu \geqslant 0$ in all of $\mathbb{R}^n$ which is an issue!

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Proposition (Fall, Jarohs, '15)

Let $\Omega \subset \mathbb{R}^n_+$ be an open, bounded set and suppose that u satisfies: $(-\Delta)^s v = 0$ in Ω, $v \geqslant 0$ in $\mathbb{R}^n_+ \setminus \Omega$, and v is antisymmetric with respect to $\partial \mathbb{R}^n_+$. Then $v \geqslant 0$ in Ω.

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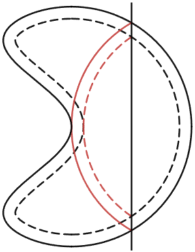
- Case 1: There exists $p \in (G'_\lambda \cap \partial G) \setminus T_\lambda \subset \Sigma'_\lambda$ since ∂G is a parallel to $\partial\Omega$. But u is constant on ∂G, so we have

$$v_\lambda(p) = u(p) - u(\text{reflection of } p) = 0$$

which contradicts assumption.



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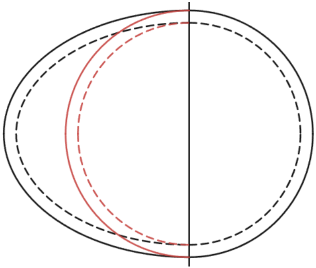
- Case 2: There exists $q \in T_\lambda \cap \partial G$ such that e is tangent to ∂G at q. Since u is a constant, we have

$$\frac{\partial v_\lambda}{\partial e}(q) = 0.$$



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This contradicts the following Hopf-type lemma:

Lemma (Dipierro, Poggesi, T, Valdinoci,'22)

Suppose that $c \in L^\infty(B_1^+)$, $u \in C^2(B_1) \cap L^\infty(\mathbb{R}^n)$ is antisymmetric with respect to $\{x_1 = 0\}$, and satisfies $(-\Delta)^s u + cu \geqslant 0$ in B_1^+, $u(x) \geqslant 0$ in $\mathbb{R}^n_+$, $u > 0$ in B_1^+. Then

$$\partial_1 u(0) > 0.$$

Stability

Question: Suppose that, instead of well-posed PDE + overdetermined condition, we have well-posed PDE + "almost" overdetermined condition. Does this mean the region Ω is "almost" a ball?

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We measure how close u is to being constant on ∂G via

$$[u]_{\partial G} := \sup_{\substack{x,y \in \partial G \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|}$$

and we measure how close Ω is to being a ball via

$$\rho(\Omega) := \inf \left\{ R - r \text{ s.t. } p \in \Omega \text{ and } B_r(p) \subset \Omega \subset B_R(p) \right\}.$$

Some literature

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- [Aftalion, Busca, Reichel, '99] Serrin's problem (with semilinearity):

$$\rho(\Omega) \leqslant C \Big| \log \|\partial_\nu u - c\|_{C^1(\partial\Omega)} \Big|^{-1/n}$$

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- [Ciraolo, Dipierro, Poggesi, Pollastro, Valdinoci, '22] Nonlocal parallel surface problem with $f = 1$:

$$\rho(\Omega) \leqslant C[u]_{\partial G}^{\frac{1}{s+2}}$$

Stability

In an upcoming work with Dipierro, Poggesi, and Valdinoci:

Theorem

Let G be an open bounded subset of $\mathbb{R}^n$ and $\Omega := G + B_R$ for some $R > 0$ be such that $\partial\Omega$ is C^2. Moreover, let $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ with $f(0) \geqslant 0$. If $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^n)$ is non-negative and satisfies

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

then

$$\rho(\Omega) \leqslant C[u]_{\partial G}^{\frac{1}{s+2}}.$$

Open problem: the optimal exponent

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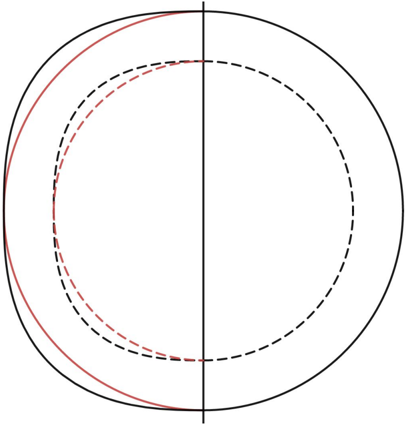
- For G_ε such that

$$G_\varepsilon + B_{1/2} = \left\{ \frac{x_1^2}{(1+\varepsilon)^2} + |x'|^2 = 1 \right\} =: \Omega_\varepsilon,$$

  we have that $\rho(\Omega_\varepsilon) = \varepsilon$ and $[u_\varepsilon]_{\partial G_\varepsilon} \simeq \varepsilon$. This suggests that $\alpha = 1$ (as in the local case).
- Nonlocality creates difficulties because it sees 'mass that is far away'.

Open problem: the optimal exponent

Suppose that $f \equiv 1$ and $[u]_{\partial G}$ is small, so that Ω is uniformly close to a ball, say B_1. Moreover, consider the situation when the reflected region is precisely B_1 and the critical plane in the direction $e = e_1$ is $\{x_1 = 0\}$:



- The reflected function v_λ (at the critical time) is s-harmonic in B_1, so, by the nonlocal Poisson representation formula,

$$v_\lambda(x) = \int_{\Omega_- \setminus B_1^-} \left(\frac{1-|x|^2}{|y|^2-1} \right)^s \left(\frac{1}{|x-y|^n} - \frac{1}{|(-x_1, x') - y|^n} \right) u(y)\, dy$$

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for all $x \in B_1$.

- Using that $G \subset\subset B_1$ (Ω is uniformly close to B_1) and regularity theory for the fractional Laplacian, one can show that

$$\int_{\Omega_- \setminus B_1^-} \frac{\delta_{\partial\Omega}^s}{\delta_{\partial B_1}^s}\, dy \leqslant C[u]_{\partial G}$$

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- If one can show that $\int_{\Omega_- \setminus B_1^-} \frac{\delta_{\partial\Omega}^s}{\delta_{\partial B_1}^s}\, dy \simeq \rho(\Omega)$ as $[u]_{\partial G} \to 0^+$ then we are done (kind of...), but this requires fine estimates up to the boundary!

Thank you for listening!