# Concavity properties of solutions to Robin problems

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- The study of concavity properties of solutions to PDEs is a classical field. Most known results concern problems under Dirichlet boundary conditions

$$u = 0 \text{ on } \partial\Omega$$

[Kawohl '86, Guan-Ma '05]

- The study of Robin boundary value problems is an emerging field, with plenty of open questions

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \text{ on } \partial\Omega$$

[Bucur-Freitas-Kennedy' 17, Laugesen '19]

- What about the "crossway" of these two subjects?

## *OUTLINE*

I. *Introduction: background, motivation, and statement of the problem*

II. *The negative result for small $\beta$ by Andrews-Clutterbuck-Hauer*

III. *A positive result for large $\beta$, with some hints on the proof*

IV. *Open questions*

## Background

- Two classical results from the 70's

[Makar Limanov '71]: Let $\Omega$ be a convex domain in $\mathbb{R}^2$.

Let $u$ be the Dirichlet torsion function of $\Omega$, that is the unique solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Then $u^{1/2}$ is concave.

[Brascamp-Lieb '76]: Let $\Omega$ be a convex domain in $\mathbb{R}^N$.

Let $u$ be the Dirichlet ground state of $\Omega$, that is a positive solution to

$$\begin{cases} -\Delta u = \lambda_1^D(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Then $\log u$ is concave.

- New methods for a systematic approach from the 80's

  [Korevaar '83]: concavity maximum principle

  [Caffarelli-Friedman '85]: continuity method

  [Caffarelli-Spruck '82, Kennington '85, Kawohl '86, Korevaar-Lewis '87 ]

  In particular, the results of the previous slide hold in any space dimension and as strict concavity inequalities

- Progress to fully nonlinear PDEs from the 90's & 00's

$$F(x, u, Du, D^2 u) = 0$$

  [Alvarez-Laszry-Lions '97]: concavity under structure conditions on $F$

  [Caffarelli-Guan-Ma '07]: constant rank theorem for viscosity solutions

- Refinements of concavity from the 10's

[Ma-Shi-Ye '12, Henrot-Nitsch-Salani-Trombetti '18]

[Andrews-Clutterbuck '11]: the Dirichlet ground state $u$ of a convex domain $\Omega$ in $\mathbb{R}^N$ with diameter $d$ satisfies the refined concavity estimate

$$\langle \nabla \log u(y) - \nabla \log u(x), \frac{y-x}{\|y-x\|} \rangle \leq -2\frac{\pi}{d}\tan\Big(\frac{\pi}{d}\frac{\|x-y\|}{2}\Big) .$$

**Consequence**: proof of the gap conjecture

$$\lambda_2^D(\Omega) - \lambda_1^D(\Omega) \geq \frac{3\pi^2}{d^2} .$$

[Van den Berg '83, Ashbaugh-Benguria '89, Yau ' 86]

What about the Robin spectral gap?

$$\lambda_2^\beta(\Omega) - \lambda_1^\beta(\Omega) \quad ?\geq? \quad \gamma_\beta \,,$$

with $\gamma_\beta$ increasing from $\frac{\pi^2}{d^2}$ to $3\frac{\pi^2}{d^2}$ as $\beta$ goes from 0 to $+\infty$

[Andrews-Clutterbuck-Hauer '20]

Problem(s):

- Is the Robin ground state of a convex domain $\Omega$ log-concave?

$$\begin{cases} -\Delta u = \lambda_1^\beta(\Omega)\, u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta\, u = 0 & \text{on } \partial\Omega, \end{cases}$$

An affirmative answer would give the gap inequality with $\gamma_\beta = \frac{\pi^2}{d^2}$.

Explicit computations for balls and rectangles suggest a positive answer.

- Is the Robin torsion function of a convex domain $\Omega$ $(\frac{1}{2})$-concave?

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta\, u = 0 & \text{on } \partial\Omega. \end{cases}$$

... SURPRISE!

**Tweet**

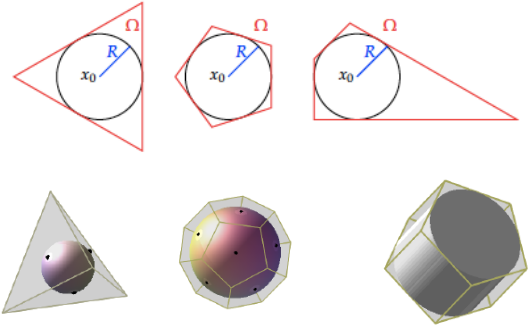
**Julie Clutterbuck** @ClutterbuckJ · Mar 4, 2018 ···

New(ish) paper: it turns out that with Robin boundary conditions, the ground state need not be log-concave. Unlike the Dirichlet case.
arxiv.org/pdf/1711.02779

## The result involves the geometry of the domain!

Let $\Omega$ be a convex polyhedron in $\mathbb{R}^N$.

- $\Omega$ is a circumsolid if there is a ball which is tangent to all its faces.
- $\Omega$ is a product of circumsolids if it is the cartesian product of circumsolids contained into orthogonal subspaces of $\mathbb{R}^N$



**Theorem 1** [Andrews-Clutterbuck-Hauer '20]

Let $\Omega$ be a convex polyhedral domain in $\mathbb{R}^N$, $N \geq 2$.

If $\Omega$ is NOT a product of circumsolids, then for sufficiently small $\beta$ the Robin ground state $u^\beta$ is NOT log-concave.

- Adding more boundary regularity there is no hope to avoid non-concavity
- Strengthened hypotheses yield nonconvex level sets for sufficiently small $\beta$

- Proof strategy: perturbation from the Neumann case

$$\log u^{\beta} = \beta v + o(\beta),$$

where $v$ is the unique solution to

$$\begin{cases} -\Delta v = \frac{|\partial\Omega|}{|\Omega|} & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = -1 & \text{on } \partial\Omega \end{cases}$$

The log-concavity of $u^{\beta}$ relates directly to the concavity properties of $v$:

$v$ concave $\Leftrightarrow$ $v \in C^2(\overline{\Omega})$ $\Leftrightarrow$ $v$ quadratic $\Leftrightarrow$ $\Omega =$ product of circumsolids

Conjecture for large $\beta$ [Andrews-Clutterbuck-Hauer '20]

For a given bounded convex domain $\Omega \subset \mathbb{R}^N$,
is the Robin ground state $u^\beta$ log-concave for $\beta \geq \beta^*$?

If affirmative, how $\beta^*$ depends on $N$ and on the geometry of the domain $\Omega$?

**Theorem 2** [Crasta-F. '21]
Let $\Omega \subset \mathbb{R}^N$ be a uniformly convex open set of class $C^m$, with $[m - \frac{N}{2}] \geq 4$.
There exists a positive threshold $\beta^*$ such that, for $\beta \geq \beta^*$, the Robin ground state of $\Omega$ is strictly log-concave.
Moreover, $\beta^*$ is uniform for convex domains of class $C^m$ satisfying

$$d(\Omega) \leq \overline{d}\,, \qquad \delta_m(\Omega) \leq \overline{\delta}\,, \qquad \kappa_{min}(\Omega) \geq \overline{\kappa}$$

- $d(\Omega) :=$ the diameter of $\Omega$;
- $\delta_m(\Omega) := \sum_{|\alpha| \leq m} \max_{x_0 \in \partial\Omega} |\partial^\alpha \varphi_{x_0}(0)|$ (max of "higher order curvatures");
- $\kappa_{min}(\Omega) := \min_{x \in \partial\Omega} \min_{i=1,\dots,N-1} \{\kappa_i(x)\}$ ($\kappa_i =$ principal curvatures).

## A glance at the proof

### The approach via continuity method:

Let $\Omega$ be given, let $\Omega_0$ be a ball, and for $t \in [0,1]$ set

$$\Omega_t = (1-t)\Omega_0 + t\Omega, \qquad v_t^\beta = -\log u_t^\beta$$

<u>**Goal**</u>: **$\nabla^2 v_1^\beta$ is positive definite in $\Omega$ (for large $\beta$).**

By contradiction: assume this is *false*. Since $\nabla^2 v_0^\beta$ is positive definite in $\Omega_0$,

$\exists s \in (0,1)$ : $\nabla^2 v_s^\beta$ is positive semidefinite, but *not* positive definite in $\Omega_s$.

Since

$$\Delta v_s^\beta = \lambda_1^\beta(\Omega_s) + |\nabla v_s^\beta|^2 ,$$

$\nabla^2 v_s^\beta$ has *constant rank* in $\Omega_s$ $\quad\Rightarrow\quad$ contradiction

**provided $\nabla^2 v_s^\beta$ is positive definite close to $\partial\Omega_s$ (for large $\beta$).**

## Strict convexity near the boundary

$$\langle \nabla^2(-\log u)\cdot\eta,\eta\rangle = \underbrace{-\frac{1}{u}\langle \nabla^2 u\cdot\eta,\eta\rangle}_{(1)} + \underbrace{\frac{|\nabla u\cdot\eta|^2}{u^2}}_{(2)} \qquad \forall \eta \in S^{N-1}.$$

- Dirichlet case

$$\begin{cases} \text{Tangential directions}: & (2)=0, \quad (1)>0 \quad \left(\langle \nabla^2 u^D\cdot\tau_i,\tau_i\rangle = -\kappa_i|\nabla u^D|\right) \\ \text{Normal direction}: & (1)=\ o(2), \quad (2)>0 \end{cases}$$

- Robin case:

For large $\beta$, the idea is to treat it as a **perturbation** of the Dirichlet case:

$$\nabla^2(-\log u^\beta) = \nabla^2(-\log u^D) + \nabla^2(-\log u^\beta + \log u^D)$$

Needs: a strong convergence result + a control on the concavity threshold.

## Global regularity and asymptotic estimates as $\beta \to +\infty$

**Theorem 3** [Crasta-F. '21]

For $\Omega \in C^m$, the Robin ground state $u^\beta$ satisfies

$$u^\beta \in H^m(\Omega).$$

If in addition $[m - \frac{N}{2}] \geq 4$, it holds

$$\|u^\beta - u^D\|_{C^{2,\theta}(\overline{\Omega})} \leq \frac{M}{\beta}, \qquad \text{with } M = M(d(\Omega)_\uparrow, \delta_m(\Omega)_\uparrow, \lambda_1^D(\Omega)_\uparrow)$$

$$|\lambda_k^\beta(\Omega) - \lambda_k^D(\Omega)| \leq \frac{\Lambda_k}{\beta}, \qquad \text{with } \Lambda_k = \Lambda_k(\delta_2(\Omega)_\uparrow, \lambda_k^D(\Omega)_\uparrow)$$

- cf. [Filinovskiy '14]
- Convexity is not needed here
- Byproduct: lower bound for the Robin gap of convex $C^2$ domains

$$\lambda_2^\beta(\Omega) - \lambda_1^\beta(\Omega) \geq \frac{3\pi^2}{d(\Omega)^2} - \frac{1}{\beta}\sqrt{6}(1 + 2\sqrt{N}\,\kappa_{max})(\lambda_2^D + 1)^2.$$

## Tracking the concavity threshold

By exploiting the previous convergence result, we obtain

$$\langle \nabla^2(-\log u^\beta)(x)\eta,\eta\rangle > 0 \qquad \forall x \in \mathscr{U}(\partial\Omega),\ \eta \in S^{N-1},\ \beta \geq \beta^*$$

with

$$\beta^* = \beta^*(d(\Omega)_\uparrow, \delta_m(\Omega)_\uparrow, \kappa_{min}(\Omega)_\downarrow, \lambda_1^D(\Omega)_\uparrow, q(\Omega)_\downarrow)$$

where $q(\Omega) = \min_{\partial\Omega} |\nabla u^D|$.

**Crucial question**: is the threshold $\beta^*$ **uniform** in the family of sets

$$\Omega_t = [(1-t)\Omega_0 + t\Omega] \qquad \text{for } t \in [0,1] \quad ??$$

**Yes!** The sets $\Omega_t$ are of class $C^m$ [Ghomi '12], and

- **red quantities are bounded (with ↑ from above, with ↓ from below)**
- **blue quantities are controlled by the red ones.**

**Theorem 4** [Crasta-F. '21]

Let $u^D$ be the first Dirichlet Laplacian eigenfunction of a convex set $\Omega \subset \mathbb{R}^N$ satisfying a uniform interior sphere condition.

Then $u^D$ satisfies the following boundary gradient estimate:

$$\min_{\partial\Omega} |\nabla u^D| \geq \frac{C_2}{\rho} (\max_{\overline{\Omega}} u^D) \, C_1^{-d(\sqrt{\lambda_1^D}/2+2\sqrt{N}/\sigma)},$$

where $C_1$, $C_2$ are dimensional constants, while

- $d = d(\Omega)$ is the diameter of $\Omega$
- $r = r(\Omega)$ is the inradius of $\Omega$
- $\sigma := \min\left\{\frac{\rho}{2}, C_0 r\right\}$, with $\rho=$ radius of interior sphere, and $C_0 = \sqrt{\frac{0.833}{\lambda_1^D(B_1)}}$

**Proof** [thanks to David Jerison!]

Use a Harnack chain of balls, and an estimate for the location of the hot spot $\overline{x}$:

$\mathrm{dist}(\overline{x}, \partial\Omega) \geq C_0 \, r(\Omega)$ [Biswas-Lörinczi '19] □

**Theorem 5** [Crasta-F. '21]
Let $\Omega \subset \mathbb{R}^N$ be a uniformly convex open set of class $C^m$, with $[m - \frac{N}{2}] \geq 4$.
There exists a positive threshold $\beta^{**}$ such that, for $\beta \geq \beta^{**}$, the Robin torsion function of $\Omega$ is strictly (1/2)-concave.
Moreover, $\beta^{**}$ is uniform for convex domains of class $C^m$ satisfying

$$d(\Omega) \leq \overline{d}\,, \qquad \delta_m(\Omega) \leq \overline{\delta}\,, \qquad \kappa_{min}(\Omega) \geq \overline{\kappa}$$

# IV. Open questions

- Can the regularity assumptions of our results be removed or weakened?
- What happens in the plane for triangles or circular sectors?
- Is it possible to characterize convex domains on which Robin solutions enjoy concavity properties for all values of the parameter?
- What about Brunn-Minkowski type inequalities for Robin energies?

MANY THANKS FOR YOUR ATTENTION!