Asymptotic decay for semilinear wave equation

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Beijing International Center for Mathematical Research

Asia-Pacific Analysis and PDE seminar, Jul.06, 2020

Semilinear wave equations

Consider the Cauchy problem to the wave equation

$$\begin{cases} \Box\phi = -\partial_t^2\phi + \Delta\phi = \mu|\phi|^{p-1}\phi, \\ \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(x) \end{cases} \tag{1}$$

in $\mathbb{R}^{1+d}$. The energy

$$E[\phi](t) = \int |\partial_t\phi|^2 + |\nabla\phi|^2 + \frac{2\mu}{p+1}|\phi|^{p+1}dx$$

is conserved for sufficiently smooth solution.

- *Focusing*, $\mu = -1$;
- *Defocusing*, $\mu = 1$.
- Scaling symmetry

$$\phi_\lambda(t,x) = \lambda^{\frac{2}{p-1}}\phi(\lambda t, \lambda x)$$

Criticality in terms of the power p

- Critical in $\dot{H}^{s_p}$

$$s_p = \frac{d}{2} - \frac{2}{p-1}.$$

- $1 < p < 1 + \frac{4}{d-2}$, energy subcritical, local well-posedness;
- $p = 1 + \frac{4}{d-2}$, energy critical, existence of local solution;
- $p > 1 + \frac{4}{d-2}$, energy supcritical, nothing too much is known: small data global solution, existence of global solution with large critical Sobolev norm (Krieger-Schlag 20', Luk-Oh-Y. 18', Soffer 18'. ect.), finite time blow up for defocusing systems(Tao 16'). Recent breakthrough blow up results for defocusing NLS by Merle-Raphael-Rodnianski-Szeftel.

Finite time blow up for the focusing case

For the focusing case, ODE type blow up in finite time can happen. Indeed the following function

$$v(t) = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} (T-t)^{-\frac{2}{p-1}}$$

verifies the equation $\partial_{tt}^2 v(t) = v(t)^p$. Now by choosing a cut-off function $\varphi(x)$ which is equal to 1 when $|x| \leq 2T$, we see that the solution with data $(\varphi(x)v(0), \varphi(x)\partial_t v(0))$ must blow up in finite time.

Focusing Energy Critical

Focusing $\mu = -1$, energy critical $p = 1 + \frac{4}{d-2}$, existence of ground state

$$\Delta W(x) + |W|^{\frac{4}{d-2}} W(x) = 0, \quad W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}$$

- Kenig-Merle 08': global existence and scattering with data under the ground state for $3 \leq d \leq 5$.
- Kenig, Merle, Liu, Duyckaerts, Jia, Lawrie ect.: soliton resolution conjecture.

Defocusing Energy Critical

- Struwe 89', $d = 3$, global solution with spherical symmetry;
- Grillakis 90', $3 \leq d \leq 5$, global regularity of the solution. This result has been extended to $d \leq 9$ by Shatah-Struwe 93', Kapitanski 94';
- Kapitanski 90', also showed that the existence of unique global weak solution in energy space for all dimension.
- Shatah-Struwe 94', finally addressed the global well-posedness in energy space for all dimension.
- Bahouri-Gérard 98', scattering by observing that the potential energy decays to zero.

Defocusing Energy Subcritical

- Ginibre-Velo 85', global well-posedness in energy space.
- $d = 1$, Lindblad-Tao 12', averaged decay

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|\phi(t,x)\|_{L_x^\infty} dx = 0.$$

  In particular the solution asymptotically does not behave like linear wave.
- Pointwise estimate, $2 \leq d \leq 3$;
- Scattering theory, consists of constructing a wave operator and proving asymptotic completeness.

Pointwise decay

- Strauss 68', $d = 3$, superconformal case $3 \leq p < 5$

$$|\phi| \leq Ct^{\epsilon - 1}.$$

- Wahl 72', improved to t^{-1} for $3 < p < 5$ and $t^{-1} \ln t$ for $p = 3$.
- Bieli-Szpak 10', improved sharp decay

$$|\phi(t, x)| \leq C(1 + t + |x|)^{-1}(1 + |t - |x||)^{2-p}.$$

- Pecher 82', $2.3 < \frac{1+\sqrt{13}}{2} < p < 3$, then

$$|\phi(t, x)| \leq Ct^{\frac{6+2p-2p^2}{3+p}} + \epsilon.$$

- Glassey-Pecher 82', $d = 2$

$$|\phi(t, x)| \leq \begin{cases} t^{-\frac{1}{2}}, & p > 5; \\ t^{-\frac{p-1}{p+3} + \epsilon}, & \frac{3+\sqrt{33}}{2} < p \leq 5; \\ t^{\frac{7+2p-p^2}{p+3} + \epsilon}, & 1 + \sqrt{8} < p \leq \frac{3+\sqrt{33}}{2}. \end{cases}$$

Complete scattering theory

Constructing a one to one map in weighted energy space:

- Ginibre-Velo 87', $d \geq 2$, $1 + \frac{4}{d-1} \leq p < 1 + \frac{4}{d-2}$, in weighted energy space (or conformal energy space) with $\gamma = 2$

$$\mathcal{E}_\gamma[\phi] = \int_{\mathbb{R}^d} (1 + |x|)^\gamma (|\phi_1|^2 + |\nabla \phi_0|^2 + \frac{2}{p+1}|\phi|^{p+1}) dx.$$

- Baez-Segal-Zhou 90', $d = 3$, $p = 3$, still in conformal energy space, using conformal method.
- Hidano 01', 03', extended to

$$3 \leq d \leq 5, \quad p > \frac{d + 2 + \sqrt{d^2 + 8d}}{2(d-1)},$$

covers part of subconformal cases. Similar result also holds in $d = 6$ and $d = 7$ but with spherical symmetry.

Asymptotic completeness in other space

Compare the solution with linear waves at time infinity.

- Asymptotic completeness in the the above mentioned results

$$\lim_{t\to\infty}\|\Gamma^\alpha\phi(t,x)-\Gamma^\alpha\phi^+(t,x)\|_{L_x^2}=0,\quad \forall|\alpha|\leq 1,$$
$$\Gamma\in\{\partial_\mu,\Omega_{\mu\nu}=x_\mu\partial_\mu-x_\nu\partial_\mu,S=t\partial_t+r\partial_r\}$$

- Pecher, scatters in energy space $\dot{H}^1$ with

$$d=3,\quad p>2.7005,\qquad \text{or } d=2,\quad p>4.15.$$

- Shen 17', $d=3$, $3\leq p<5$ with spherical symmetry, scatters in $\dot{H}^{s_p}$ for data in $\mathcal{E}_{1+\epsilon}[\phi]$. This recently was greatly improved by Dodson for data bounded in the critical Sobolev space $\dot{H}^{s_p}$.

Global behavior in higher dimension

Theorem (Y. 2019)

For $d \geq 3$, the solution verifies the following asymptotical decay properties:

- *For $1 < p \leq \frac{d+2}{d-2}$, an integrated local energy decay estimate*

$$\iint_{\mathbb{R}^{1+d}} \frac{|\partial\phi|^2 + |(1+r)^{-1}\phi|^2}{(1+r)^{1+\epsilon}} + \frac{|\phi|^{p+1} + |\nabla\!\!\!/\phi|^2}{r} dxdt \leq C\mathcal{E}_0[\phi]$$

- *For $\frac{d+1}{d-1} < p \leq \frac{d+2}{d-2}$ and $1 < \gamma_0 < \min\{2, \frac{1}{2}(p-1)(d-1)\}$,*

$$E[\phi](\Sigma_u) + \iint_{D_u} \frac{|\partial\phi|^2 + |\phi|^{p+1}}{(1+r)^{1+\epsilon}} dxdt \leq C u_+^{-\gamma_0}\mathcal{E}_{\gamma_0}[\phi],$$

$$\iint_{\mathbb{R}^{1+d}} v_+^{\gamma_0-\epsilon-1}|\phi|^{p+1} dxdt \leq C\mathcal{E}_{\gamma_0}[\phi].$$

Here $u = t - r$, $u_+ = 1 + |u|$, $v = t + r$, $v_+ = 1 + v$.

Scattering in higher dimension

Corollary (Y. 2019)

Assume that $d \geq 3$ *and*

$$\frac{1+\sqrt{d^2+4d-4}}{d-1} < p < \frac{d+2}{d-2},$$
$$\max\{\frac{4}{p-1} - d + 2, 1\} < \gamma_0 < \min\{\frac{1}{2}(p-1)(d-1), 2\}$$

then the solution is uniformly bounded

$$\|\phi\|_{L_{t,x}^{\frac{(d+1)(p-1)}{2}}} \leq C(p, d, \gamma_0, \mathcal{E}_{\gamma_0}[\phi])$$

As a consequence , there exist pairs $\phi_0^{\pm} \in \dot{H}_x^{s_p} \cap \dot{H}_x^1$ *and* $\phi_1^{\pm} \in \dot{H}_x^{s_p-1} \cap L_x^2$ *such that for all* $s_p \leq s \leq 1$

$$\lim_{t \to \pm\infty} \|(\phi(t,x), \partial_t \phi(t,x)) - \mathbf{L}(t)(\phi_0^{\pm}(x), \phi_1^{\pm}(x))\|_{\dot{H}_x^s \times \dot{H}_x^{s-1}} = 0.$$

Pointwise decay in dimension 3

Theorem (Y. 2019)

In $\mathbb{R}^{1+3}$, the solution verifies the following pointwise decay estimates

- *For the case when*

$$\frac{1+\sqrt{17}}{2} < p < 5, \quad \max\{\frac{4}{p-1} - 1, 1\} < \gamma_0 < \min\{p-1, 2\},$$

  then

$$|\phi(t,x)| \leq C(1 + \mathcal{E}_{1,\gamma_0}[\phi])^{\frac{p-1}{2}} (1 + t + |x|)^{-1} (1 + ||x| - t|)^{-\frac{\gamma_0 - 1}{2}};$$

- *Otherwise if* $2 < p \leq \frac{1+\sqrt{17}}{2}$ *and* $1 < \gamma_0 < p - 1$*, then*

$$|\phi(t,x)| \leq C\sqrt{\mathcal{E}_{1,\gamma_0}[\phi]} (1 + t + |x|)^{-\frac{3+(p-2)^2}{(p+1)(5-p)}\gamma_0} (1 + ||x| - t|)^{-\frac{\gamma_0}{p+1}}$$

Improved scattering in energy space in dimension 3

The above pointwise decay estimate for the solution can be used to show the scattering in energy space with improved lower bound of p.

Corollary (Y. 2019)

For $p > 2.3542$ and initial data bounded in $\mathcal{E}_{1,p-1}[\phi]$, the solution ϕ is uniformly bounded in the following mixed spacetime norm

$$\|\phi\|_{L_t^p L_x^{2p}} < \infty.$$

Consequently the solution scatters in energy space, that is, there exists pairs $(\phi_0^{\pm}(x), \phi_1^{\pm}(x))$ such that

$$\lim_{t\to\pm\infty} \|\partial\phi(t,x) - \partial\mathbf{L}(t)(\phi_0^{\pm}(x), \phi_1^{\pm}(x))\|_{L_x^2} = 0.$$

Pointwise bound in dimension 3 with small p

The above results are based on the vector field method originally introduced by Dafermos and Rodnianski, which however fails in lower dimension or for the case in dimension 3 but with small power $p \leq 2$. By introducing new vector fields as multipliers, we are able to derive quantitative pointwise bound for the solution for all $p > 1$.

Theorem (Wei-Y.)

For all $1 < p \leq 2$, the solution ϕ verifies the following pointwise bound

$$|\phi(t,x)| \leq C\sqrt{\mathcal{E}_{1,2}[\phi]}(1+t+|x|)^{\frac{\epsilon-(p-1)^2}{p+1}}(1+|t-|x||)^{\frac{3-2p+\epsilon}{p+1}}.$$

for some constant C depending only on $\epsilon > 0$ and p. As a consequence, the solution decays uniformly in time

$$|\phi(t,x)| \leq C\sqrt{\mathcal{E}_{1,2}[\phi]}(1+t)^{\frac{\epsilon+2-p^2}{p+1}}$$

when $p > \sqrt{2}$.

Asymptotic decay in dimension 1

As conjectured by Lindblad and Tao, in dimension $d = 1$, the solution should decay in time with an inverse polynomial rate. We give this conjecture an affirmative answer.

Theorem (Wei-Y. 2020)

In $\mathbb{R}^{1+1}$ and for all $p > 1$, the solution ϕ decays in the following sense

$$|\phi(t,x)| \leq C(1+t)^{-\frac{p-1}{(p+1)^2+4}}$$

for some constant C depending only on p and the initial weighted energy $\sqrt{\mathcal{E}_{0,1}[\phi]}$.

Asymptotic decay in dimension 2

For the space dimension two case, we show that

Theorem (Wei-Y. 2020)

For the subconformal case $1 < p \leq 5$*, we have the potential energy decay*

$$\int_{\mathbb{R}^2} |\phi(t,x)|^{p+1} dx \leq C\mathcal{E}_{0,2}[\phi](1+t)^{-\frac{p-1}{2}}, \quad \forall t \geq 0$$

as well as the pointwise decay estimates

$$|\phi(t,x)| \leq \begin{cases} C(1+t)^{-\frac{1}{2}}, & \frac{11}{3} < p \leq 5; \\ C_\epsilon(1+t)^{-\frac{p-1}{8}+\epsilon}, & 1 < p \leq \frac{11}{3}. \end{cases}$$

As a consequence the solution scatters in the critical Sobolev space when $p > 1+\sqrt{8}$ *and scatters in energy space when* $p > 2\sqrt{5}-1$.

Conformal energy

All the previous results heavily rely on the following conformal energy identity obtained by using the conformal vector field $K = (t^2 + r^2)\partial_t + 2tr\partial_r$ as multiplier

$$\begin{aligned}
&Q_0(t) + \frac{2}{p+1}\int (t^2 + r^2)|\phi(t,x)|^{p+1}dx \\
&+ \frac{d-1}{p+1}(p - \frac{d+3}{d-1})\int_s^t 2\tau \int |\phi(\tau,x)|^{p+1}dxd\tau \\
&= Q_0(s) + \frac{2}{p+1}\int (s^2 + |x|^2)|\phi(s,x)|^{p+1}dx
\end{aligned}$$

where

$$Q_0(t) = \sum_{0\le\mu,\nu\le d}\int |\Omega_{\mu\nu}\phi|^2 + |S\phi + (d-1)\phi|^2 dx$$

The superconformal case

For the superconformal case when

$$p \geq \frac{d+3}{d-1}$$

the left hand side of the previous conformal energy identity is nonnegative.

$$Q_0(t) + \int (t^2 + r^2)|\phi(t,x)|^{p+1} dx \leq \mathcal{E}_2[\phi].$$

This time decay is sufficient to conclude the scattering and pointwise (for $2 \leq d \leq 3$) properties of the solution.

The subconformal case

Key observation to go beyond the conformal power ($p < \frac{d+3}{d-1}$) is to use Gronwall's inequality. Define

$$G(t) = t^2 \int |\phi(t,x)|^{p+1} dx$$

The previous conformal energy identity implies that

$$G(t) \leq G(0) + (d+3-p(d-1)) \int_0^t \tau^{-1} G(\tau) d\tau.$$

Thus

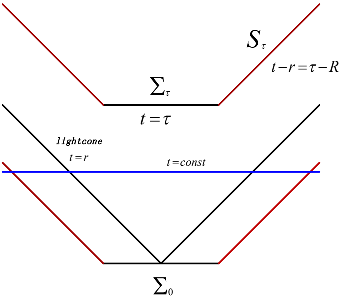
$$G(t) \leq G(0) t^{d+3-p(d-1)}.$$

The leads to the time decay of the potential energy for p close to the conformal power

$$\int_{\mathbb{R}^d} |\phi(t,x)|^{p+1} dx \leq C\mathcal{E}_2[\phi](1+t)^{d+1-p(d-1)}.$$

Dafermos-Rodnianski's new approach

The foliation



together with multipliers

$$f(r)\partial_r, \quad \partial_t, \quad r^p(\partial_t+\partial_r)$$

to derive the integrated local energy decay, the classical energy estimate and a hierarchy of r-weighted energy estimates. Using a pigeon hole argument, one can derive the energy flux decay.

The r-weighted energy identity for semilinear wave equation

We have the following energy identity for $0 \leq \gamma \leq 2$.

$$\iint_{D_u} r^{\gamma-d}(\gamma|L\psi|^2 + (2-\gamma)(|\nabla\!\!\!/\psi|^2 + c_d r^{-2}|\psi|^2)) + c_{\gamma,d} r^{\gamma-1}|\phi|^{p+1}$$
$$+ \int_{H_u} 2r^\gamma |L\psi|^2 dv d\omega + \int_{\mathcal{I}_u} 2r^\gamma (|\nabla\!\!\!/\psi|^2 + \frac{2}{p+1}|\phi|^{p+1} r^{d-1} + c_d r^{-2}|\psi|^2) du d\omega$$
$$= \int_{\{t=0,|x|\geq 2|u|\}} r^\gamma (|L\psi|^2 + |\nabla\!\!\!/\psi|^2 + \frac{2}{p+1}|\phi|^{p+1} r^{d-1} + c_d r^{-2}|\psi|^2) dr d\omega$$

Here $\psi = r^{\frac{d-1}{2}}\phi$, $L = \partial_t + \partial_r$, $c_d = \frac{(d-1)(d-3)}{4}$ and

$$c_{\gamma,d} = d - 1 - \frac{2\gamma + 2d - 2}{p+1} = \frac{(d-1)(p-1) - 2\gamma}{p+1}.$$

Energy flux decay

This new method enables us to derive the energy flux decay

$$\int_{H_u} |L\phi|^2 + |\nabla\!\!\!/\phi|^2 + \frac{2}{p+1}|\phi|^{p+1} d\sigma \leq C\mathcal{E}_{\gamma_0}[\phi] u_+^{-\gamma_0}.$$

Integrate in terms of u, we obtain

$$\iint_{\mathbb{R}^{1+d}} |\phi|^{p+1} u_+^{\gamma-1} dxdt \leq C\mathcal{E}_{\gamma_0}[\phi], \quad \forall 0 < \gamma < \gamma_0.$$

Combining this with the r-weighted energy estimate

$$\iint_{\mathbb{R}^{1+d}} r^{\gamma_0 - 1}|\phi|^{p+1} dxdt \leq C\mathcal{E}_{\gamma_0}[\phi]$$

we conclude that

$$\iint_{\mathbb{R}^{1+d}} v_+^{\gamma-1}|\phi|^{p+1} dxdt \leq C\mathcal{E}_{\gamma_0}[\phi], \quad \forall 0 < \gamma < \gamma_0.$$

The improvement of the time decay of the potential energy

This new method requires

$$1 < \gamma < \frac{(d-1)(p-1)}{2}, \quad \gamma < \gamma_0, \quad \mathcal{E}_{\gamma_0}[\phi] < \infty,$$
$$p > 1 + \frac{2}{d-1} = \frac{d+1}{d-1},$$

Recall the previous time decay of the potential energy

$$\int_{\mathbb{R}^d} |\phi(t,x)|^{p+1} dx \leq C\mathcal{E}_2[\phi](1+t)^{d+1-p(d-1)}.$$

Compared to the new time decay

$$\int_{\mathbb{R}^d} |\phi(t,x)|^{p+1} dx \leq C\mathcal{E}_{\gamma_0}[\phi](1+t)^{-\gamma}.$$

For the subconformal case when $p < \frac{d+3}{d-1}$, note that

$$\frac{(d-1)(p-1)}{2} > p(d-1) - (d+1).$$

Proof for the one dimensional case

The key estimate in the work of Lindblad and Tao is the improved potential energy decay

$$\int_{t_0-T}^{t_0+T}\int_{x_0+vt-R}^{x_0+vt+R}|\phi(t,x)|^{p+1}dxdt \leq C(\sqrt{RT}+R^{-1}T), \quad \forall T \geq R > 0$$

on parallelogram, derived by using the vector field $v\partial_t + \partial_x$ as multiplier. The averaged decay estimate of the solution then follows by using the classical Rademacher differentiation theorem.

One of the key new ingredients of our proof is the new multipliers

$$\beta^{-1}(1+t-x)^{\beta}(1+t+x)^{\alpha-1}(\partial_t-\partial_x)+\alpha^{-1}(1+t-x)^{\beta-1}(1+t+x)^{\alpha}(\partial_t+\partial_x)$$

with constants α, β such that

$$\left(\frac{1}{\alpha}-1\right)\left(\frac{1}{\beta}-1\right)=\frac{4}{(p+1)^2}, \quad \frac{1}{2}\leq \alpha < 1.$$

Proof for the two dimensional case

One of the key new ideas in dimension two is to apply a new class of non-spherically symmetric vector fields

$$X = u_1^{\frac{p-1}{2}}(\partial_t - \partial_1) + u_1^{\frac{p-1}{2}-2} x_2^2(\partial_t + \partial_1) + 2u_1^{\frac{p-1}{2}-1} x_2\partial_2$$

with $u_1 = t - x_1 + 1$ as multipliers to to regions bounded by hyperplanes $\{t = x_1\}$. This enables us to derive the improved time decay of the potential energy

$$\int_{\mathbb{R}^2} |\phi(t,x)|^{p+1} dx \le C\mathcal{E}_{0,2}[\phi](1+t)^{-\frac{p-1}{2}}, \quad p \le 5.$$

The pointwise decay estimate for the solution for all $p > 1$ relies on the following Brézis-Gallouet-Wainger inequality

$$\|u\|_{L^\infty(\mathbb{R}^2)} \le C\|u\|_{H^1(\mathbb{R}^2)} \left(1 + \ln \frac{\|u\|_{H^2(\mathbb{R}^2)}}{\|u\|_{H^1(\mathbb{R}^2)}}\right)^{\frac{1}{2}}$$

Proof for the three dimensional case with small p

The proof is inspired by the method in dimension two. Instead of using spherically symmtric vector fields as multiplier, we try

$$X = u^{p-1}(\partial_t - \partial_1) + u^{p-3}(x_2^2 + x_3^2)(\partial_t + \partial_1) + 2u^{p-2}(x_2\partial_2 + x_3\partial_3)$$

with $u = t - x_1$. Applying this vector field to the backward light cone $\mathcal{N}^-(q)$ with $q = (t_0, r_0, 0, 0)$, we derive the weighted energy estimate

$$\int_{\mathcal{N}^-(q)} |t_0 - r_0|^{p-1}(1 - \frac{x_1 - r_0}{|x - x_0|})d\sigma \leq C.$$

The key point of using such non-spherically symmetric vector field is that it allows us to use the reflection symmetry $x_1 \to -x_1$ to obtain that

$$\int_{\mathcal{N}^-(q)} |t_0 + r_0|^{p-1}(1 + \frac{x_1 - r_0}{|x - x_0|})d\sigma \leq C.$$

Thank you!